UNIT-I & UNIT-II

1.1 Functions and their Limits

Function: Let A and B be two non empty sets. A rule $f: A \to B$ (read as f from A to B) is said to be a function if to each element x of A there exists a unique element y of B such that $f(x) = y$.

$\boldsymbol{y}$ is called the image of $\boldsymbol{x}$ under the map $\boldsymbol{f}$. Here $\boldsymbol{x}$ is independent variable and $\boldsymbol{y}$ is dependent variable.

There are mainly two types of functions: Explicit functions and Implicit functions. If y is clearly expressed in the terms of x directly then the function is called Explicit function. e.g. $y = x + 20$.

If y can't be expressed in the terms of x directly then the function is called Implicit function. e.g. $a\,x^2 + 2hxy + by^2 = 1$.

We may further categorized the functions according to their nature as:

Functions Types	Algebraic	Trigonometric	Inverse Trigonometric	Exponential	Logarithmic
Examples	$y = x^2 + x + 1$, $y = x^3 - 3x + 2$ etc.	$y = sinx$, $y = secx$ etc.	$y = tan^{-1}x$, $y = cos^{-1}x$ etc.	$y = e^x$, $y = 2^x$ etc.	$y = log_e x$, $y = log_5 x$ etc.

Even Function: A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x.

For example: x^2 , $x^4 + 1$, $\cos(x)$ etc.

Odd Function: A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$ for all x.

For example: x^3 , $\sin(x)$, $\tan(x)$ etc.

Periodic Function: A function $f(x)$ is said to be a periodic function if it retains same value after a certain period.

For example: $\sin(x)$, $\tan(x)$ etc.

As $\sin(x) = \sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi)$ ...

Therefore $\sin(x)$ is a periodic function with period 2π.

Some Solved Problems:

Q.1. If $f(x) = x^2 + 1$, find $f(2)$.

Sol. Given that $f(x) = x^2 + 1$ (1.1)

Put $x = 2$ in (1.1), we get

$f(2) = 2^2 + 1 = 4 + 1 = 5.$

Q.2. If $f(x) = x^3 - 1$, find $f(0)$.

Sol. Given that $f(x) = x^3 - 1$ (2.1)

Put $x = 0$ in (2.1), we get

$f(0) = 0^3 - 1 = 0 - 1 = -1.$

Q.3. If $f(x) = x^3 + 2x^2 - 3x + 1$, find $f(-1)$.

Sol. Given that $f(x) = x^3 + 2x^2 - 3x + 1$ (3.1)

Put $x = -1$ in (3.1), we get

$f(-1) = (-1)^3 + 2(-1)^2 - 3(-1) + 1 = -1 + 2 + 3 + 1 = 5.$

Q.4. If $f(x) = x^2 + x + 1$, find $f(2).f(3)$.

Sol. Given that $f(x) = x^2 + x + 1$ (4.1)

Put $x = 2$ in (4.1), we get

$f(2) = 2^2 + 2 + 1 = 4 + 2 + 1 = 7$

Again put $x = 3$ in (4.1), we get

$f(3) = 3^2 + 3 + 1 = 9 + 3 + 1 = 13$

Therefore $f(2).f(3) = 7 \times 13 = 91.$

Q.5. If $f(x) = 2x^2 - 4x + 6$, find $\frac{f(-2)}{f(1)}$.

Sol. Given that $f(x) = 2x^2 - 4x + 6$ (5.1)

Put $x = -2$ in (5.1), we get

$f(-2) = 2(-2)^2 - 4(-2) + 6 = 2(4) + 8 + 6 = 8 + 14 = 22$

Again put $x = 1$ in (5.1), we get

$f(1) = 2(1)^2 - 4(1) + 6 = 2 - 4 + 6 = 4$

Therefore $\frac{f(-2)}{f(1)} = \frac{22}{4} = 5.5$

Q.6. If $f(x) = 3x^3 - 5x + 3$, find $f(a^2)$.

Sol. Given that $f(x) = 3x^3 - 5x + 3$ (6.1)

Put $x = a^2$ in (6.1), we get

$f(a^2) = 3(a^2)^3 - 5(a^2) + 3 = 3a^6 - 5a^2 + 3$

Q.7. If $(x) = \frac{1}{1+x}$, find $f\left(\frac{1}{x}\right)$.

Sol. Given that $f(x) = \frac{1}{1+x}$ (7.1)

Replace x by $\frac{1}{x}$ in (7.1), we get

$f\left(\frac{1}{x}\right) = \frac{1}{1+\frac{1}{x}} = \frac{1}{\left(\frac{x+1}{x}\right)} = \frac{x}{x+1}$.

Q.8. If $g(x) = x^3 + 2x^2 + 5x + 10$, find $g(-2) + g(-1)$.

Sol. Given that $g(x) = x^3 + 2x^2 + 5x + 10$ (8.1)

Put $x = -2$ in (8.1), we get

$g(-2) = (-2)^3 + 2(-2)^2 + 5(-2) + 10 = -8 + 8 - 10 + 10 = 0$

Again put $x = -1$ in (8.1), we get

$g(-1) = (-1)^3 + 2(-1)^2 + 5(-1) + 10 = -1 + 2 - 5 + 10 = 6$

Therefore $g(-2) + g(-1) = 0 + 6 = 6$.

Q.9. If $h(x) = \sin(x) - x + 2$, find $h(0)$.

Sol. Given that $h(x) = \sin(x) - x + 2$ (9.1)

Put $x = 0$ in (9.1), we get

$h(0) = \sin(0) - 0 + 2 = 0 - 0 + 2 = 2$.

Q.10. If $(x) = \sqrt{2}\cos(x) - 3$, find $g\left(\frac{\pi}{4}\right)$.

Sol. Given that $g(x) = \sqrt{2}\cos(x) - 3$ (10.1)

Put $x = \frac{\pi}{4}$ in (10.1), we get

$g\left(\frac{\pi}{4}\right) = \sqrt{2}\cos\left(\frac{\pi}{4}\right) - 3 = \sqrt{2} \times \frac{1}{\sqrt{2}} - 3 = 1 - 3 = -2$.

<u>**Note**</u>:

(i) The symbol "∞" is called infinity.

(ii) $\frac{a}{0}$ is not finite $(where\ a \neq 0)$ and it is represented by ∞ .

(iii) $\frac{a}{\infty} = 0$ if $(a \neq \infty)$.

Indeterminate Forms: The following forms are called indeterminate forms:

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty^{\infty}, 0^0, \infty^0, \infty - \infty, 0 \times \infty \ etc.$$

(These forms are meaningless)

Definition of Limit: A function $f(x)$ is said to have limit l when x tends to a, if for every positive ε (however small) there exists a positive number δ such that $|f(x) - l| < \varepsilon$ for all values of x for which $0 < |x - a| < \delta$ and it is represented as

$$\lim_{x \to a} f(x) = l$$

Some basic properties on Limits:

(i) $\lim_{x \to a} K = K$ where K is some constant.

(ii) $\lim_{x \to a} K . f(x) = K . \lim_{x \to a} f(x)$ where K is some constant.

(iii) $\lim_{x \to a} \left[f(x) + g(x) \right] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

(iv) $\lim_{x \to a} \left[f(x) - g(x) \right] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

(v) $\lim_{x \to a} \left[f(x) . g(x) \right] = \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$

(vi) $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided that $\lim_{x \to a} g(x) \neq 0$

(vii) $\lim_{x \to a} \left[f(x) \right]^n = \left[\lim_{x \to a} f(x) \right]^n$

Methods of finding the limits of the functions:

1) Direct Substitution Method
2) Factorization Method etc.

Some Standard Limits Formulas:

1) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

2) $\lim_{x \to \infty} \left(1 + \frac{1}{x} \right)^x = e$

3) $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$

4) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log_e a$

5) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=\log_e e=1$

6) $\lim\limits_{x\to 0}\sin x=0$

7) $\lim\limits_{x\to 0}\tan x=0$

8) $\lim\limits_{x\to 0}\cos x=1$

9) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

10) $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$

<u>Some Solved Problems</u>:

Q.1. Evaluate $\lim\limits_{x\to 0}\left(1+2x+x^2\right)$.

Sol. $\lim\limits_{x\to 0}\left(1+2x+x^2\right)=1+2(0)+(0)^2=1+0+0=1$

Q.2. Evaluate $\lim\limits_{x\to -1}\left(1+x+x^2+x^3\right)$.

Sol. $\lim\limits_{x\to -1}\left(1+x+x^2+x^3\right)=1+(-1)+(-1)^2+(-1)^3=1-1+1-1=0$

Q.3. Evaluate $\lim\limits_{x\to 2}\dfrac{1+2x^2}{3x}$.

Sol. $\lim\limits_{x\to 2}\dfrac{1+2x^2}{3x}=\dfrac{1+2(2)^2}{3(2)}=\dfrac{1+8}{6}=\dfrac{9}{6}=1.5$

Q.4. Evaluate $\lim\limits_{x\to 3}\dfrac{x^2-9}{5}$.

Sol. $\lim\limits_{x\to 3}\dfrac{x^2-9}{5}=\dfrac{3^2-9}{5}=\dfrac{9-9}{5}=\dfrac{0}{5}=0$

Q.5. Evaluate $\lim\limits_{x\to -1}\dfrac{x^3+6}{x+1}$.

Sol. $\lim\limits_{x\to -1}\dfrac{x^3+6}{x+1}=\dfrac{(-1)^3+6}{-1+1}=\dfrac{-1+6}{0}=\dfrac{5}{0}=\infty$

Q.6. Evaluate $\lim_{x\to 2}\frac{x^3-8}{x+2}$.

Sol. $\lim_{x\to 2}\frac{x^3-8}{x+2}=\frac{2^3-8}{2+2}=\frac{8-8}{4}=\frac{0}{4}=0$

Q.7. Evaluate $\lim_{x\to 1}\frac{x^2+1}{x-1}$.

Sol. $\lim_{x\to 1}\frac{x^2+1}{x-1}=\frac{1^2+1}{1-1}=\frac{2}{0}=\infty$

Q.8. Evaluate $\lim_{x\to 0}\frac{x^3+4x^2-7x-8}{x+4}$.

Sol. $\lim_{x\to 0}\frac{x^3+4x^2-7x-8}{x+4}=\frac{(0)^3+4(0)^2-7(0)-8}{0+4}=\frac{-8}{4}=-2$

Q.9. Evaluate $\lim_{x\to 2}\frac{x^3-8}{x-2}$.

Sol. $\lim_{x\to 2}\frac{x^3-8}{x-2}=\frac{2^3-8}{2-2}=\frac{8-8}{2-2}=\frac{0}{0}$ $\left(\frac{0}{0}\,form\right)$

$$\therefore \lim_{x\to 2}\frac{x^3-8}{x-2}=\lim_{x\to 2}\frac{x^3-2^3}{x-2}$$

By factorization method

$$=\lim_{x\to 2}\frac{(x-2)\left(x^2+2^2+2x\right)}{x-2}$$

$$=\lim_{x\to 2}\left(x^2+2^2+2x\right)$$

$$=2^2+2^2+2(2)$$

$$=4+4+4=12$$

Q.10. Evaluate $\lim_{x\to 5}\frac{x^2-25}{x-5}$.

Sol. $\lim_{x\to 5}\frac{x^2-25}{x-5}=\frac{5^2-25}{5-5}=\frac{25-25}{5-5}=\frac{0}{0}$ $\left(\frac{0}{0}\,form\right)$

$$\therefore \lim_{x\to 5}\frac{x^2-25}{x-5}=\lim_{x\to 5}\frac{x^2-5^2}{x-5}$$

By factorization method

$$=\lim_{x\to 5}\frac{(x-5)(x+5)}{x-5}$$

$$=\lim_{x\to 5}(x+5)$$

$$=5+5=10$$

Q.11. Evaluate $\lim_{x\to 4}\frac{x^4-256}{x-4}$.

Sol. $\lim_{x\to 4}\frac{x^4-256}{x-4}=\frac{4^4-256}{4-4}=\frac{256-256}{4-4}=\frac{0}{0}$ $\left(\frac{0}{0}\,form\right)$

$$\therefore \lim_{x\to 4}\frac{x^4-256}{x-4}=\lim_{x\to 4}\frac{x^4-4^4}{x-4}$$

By factorization method

$$=\lim_{x\to 4}\frac{(x^2-4^2)(x^2+4^2)}{x-4}$$

$$=\lim_{x\to 4}\frac{(x-4)(x+4)(x^2+4^2)}{x-4}$$

$$=\lim_{x\to 4}(x+4)(x^2+4^2)$$

$$=(4+4)(4^2+4^2)$$

$$=8\times 32=256$$

Q.12. Evaluate $\lim_{x\to 3}\frac{x-3}{x^2-9}$.

Sol. $\lim_{x\to 3}\frac{x-3}{x^2-9}=\frac{3-3}{3^2-9}=\frac{3-3}{9-9}=\frac{0}{0}$ $\left(\frac{0}{0}\,form\right)$

$$\therefore \lim_{x\to 3}\frac{x-3}{x^2-9}=\lim_{x\to 3}\frac{x-3}{(x-3)(x+3)}$$

$$=\lim_{x\to 3}\frac{1}{(x+3)}$$

By factorization method

$$=\frac{1}{3+3}=\frac{1}{6}$$

Q.13. Evaluate $\lim_{x\to 1}\frac{x^2+x-2}{x^2-4x+3}$.

Sol. $\lim_{x\to 1}\frac{x^2+x-2}{x^2-4x+3}=\frac{1^2+1-2}{1^2-4(1)+3}=\frac{1+1-2}{1-4+3}=\frac{0}{0}$ $\left(\frac{0}{0}\,form\right)$

$\therefore \lim_{x\to 1}\frac{x^2+x-2}{x^2-4x+3}=\lim_{x\to 1}\frac{x^2+2x-x-2}{x^2-3x-x+3}$ By factorization method

$=\lim_{x\to 1}\frac{x(x+2)-1\,(x+2)}{x(x-3)-1\,(x-3)}$

$=\lim_{x\to 1}\frac{(x+2)(x-1)}{(x-3)(x-1)}$

$=\lim_{x\to 1}\frac{(x+2)}{(x-3)}=\frac{(1+2)}{(1-3)}=-\frac{3}{2}$

Q.14. Evaluate $\lim_{x\to 2}\frac{x^2+5x+6}{x^2-4}$.

Sol. $\lim_{x\to 2}\frac{x^2+5x+6}{x^2-4}=\frac{2^2+5(2)+6}{2^2-4}$

$=\frac{4+10+6}{4-4}=\frac{20}{0}=\infty$

Q.15. Evaluate $\lim_{x\to 3}\frac{x^2-9}{x^2-4x+6}$.

Sol. $\lim_{x\to 3}\frac{x^2-9}{x^2-4x+6}=\frac{3^2-9}{3^2-4(3)+6}$

$=\frac{9-9}{9-12+6}=\frac{0}{3}=0$

Q.16. Evaluate $\lim_{x\to -1}\frac{x^3+1}{x+1}$.

Sol. $\lim_{x\to -1}\frac{x^3+1}{x+1}=\frac{(-1)^3+1}{-1+1}=\frac{-1+1}{0}=\frac{0}{0}$ $\left(\frac{0}{0}\,form\right)$

$$\therefore \lim_{x\to -1}\frac{x^3+1}{x+1}=\lim_{x\to -1}\frac{(x+1)(x^2-x+1)}{x+1}$$ By factorization method

$$=\lim_{x\to -1}(x^2-x+1)$$

$$=(-1)^2-(-1)+1=1+1+1=3$$

Q.17. Evaluate $\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3}$.

Sol. $\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3}=\frac{\sqrt{3}-\sqrt{3}}{3-3}=\frac{0}{0}$ $\left(\frac{0}{0}\,form\right)$

$$\therefore \lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3}=\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{(\sqrt{x})^2-(\sqrt{3})^2}$$ By factorization method

$$=\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}$$

$$=\lim_{x\to 3}\frac{1}{(\sqrt{x}+\sqrt{3})}=\frac{1}{\sqrt{3}+\sqrt{3}}=\frac{1}{2\sqrt{3}}$$

Q.18. Evaluate $\lim_{x\to 1}\frac{x^3-1}{x-1}$.

Sol. $\lim_{x\to 1}\frac{x^3-1}{x-1}=\frac{1^3-1}{1-1}=\frac{0}{0}$ $\left(\frac{0}{0}\,form\right)$

$$\therefore \lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}$$ By factorization method

$$=\lim_{x\to 1}(x^2+x+1)$$

$$=1^2+1+1=1+1+1=3$$

Q.19. Evaluate $\lim_{x\to 2}\frac{x^3-8}{x-2}$.

Sol. $\lim_{x\to 2}\frac{x^3-8}{x-2}=\frac{2^3-8}{2-2}=\frac{8-8}{2-2}=\frac{0}{0}$ $\left(\frac{0}{0}\,form\right)$

$$\therefore \lim_{x\to 2}\frac{x^3-8}{x-2}=\lim_{x\to 2}\frac{x^3-2^3}{x-2}=3\times(2)^2=12$$

$$\left[by \quad \lim_{x \to a} \frac{x^n - a^n}{x - a} = n\,a^{n-1}, where \;\; a = 2 \;\&\; n = 3 \right]$$

Q.20. Evaluate $\lim\limits_{x \to 5} \dfrac{x^2 - 25}{x - 5}$.

Sol. $\lim\limits_{x \to 5} \dfrac{x^2 - 25}{x - 5} = \dfrac{5^2 - 25}{5 - 5} = \dfrac{25 - 25}{0} = \dfrac{0}{0}$ $\qquad \left(\dfrac{0}{0} \, form \right)$

$$\therefore \lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{x^2 - 5^2}{x - 5} = 2 \times (5)^1 = 10$$

$$\left[by \quad \lim_{x \to a} \frac{x^n - a^n}{x - a} = n\,a^{n-1}, where \;\; a = 5 \;\&\; n = 2 \right]$$

Q.21. Evaluate $\lim\limits_{x \to 3} \dfrac{x^4 - 81}{x - 3}$.

Sol. $\lim\limits_{x \to 3} \dfrac{x^4 - 81}{x - 3} = \dfrac{3^4 - 81}{3 - 3} = \dfrac{81 - 81}{0} = \dfrac{0}{0}$ $\qquad \left(\dfrac{0}{0} \, form \right)$

$$\therefore \lim_{x \to 3} \frac{x^4 - 81}{x - 3} = \lim_{x \to 3} \frac{x^4 - 3^4}{x - 3} = 4 \times (3)^3 = 108$$

$$\left[by \quad \lim_{x \to a} \frac{x^n - a^n}{x - a} = n\,a^{n-1}, where \;\; a = 3 \;\&\; n = 4 \right]$$

Q.22. Evaluate $\lim\limits_{x \to 4} \dfrac{x^3 - 64}{x - 4}$.

Sol. $\lim\limits_{x \to 4} \dfrac{x^3 - 64}{x - 4} = \dfrac{4^3 - 64}{4 - 4} = \dfrac{64 - 64}{0} = \dfrac{0}{0}$ $\qquad \left(\dfrac{0}{0} \, form \right)$

$$\therefore \lim_{x \to 4} \frac{x^3 - 64}{x - 4} = \lim_{x \to 4} \frac{x^3 - 4^3}{x - 4} = 3 \times (4)^2 = 48$$

$$\left[by \quad \lim_{x \to a} \frac{x^n - a^n}{x - a} = n\,a^{n-1}, where \;\; a = 4 \;\&\; n = 3 \right]$$

Q.23. Evaluate $\lim\limits_{x \to -1} \dfrac{x^2 - 1}{x + 1}$.

Sol. $\lim\limits_{x\to -1}\dfrac{x^2-1}{x+1}=\dfrac{(-1)^2-1}{-1+1}=\dfrac{1-1}{0}=\dfrac{0}{0}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$$\therefore\ \lim_{x\to -1}\frac{x^2-1}{x+1}=\lim_{x\to -1}\frac{x^2-(-1)^2}{x-(-1)}=2\times(-1)^1=-2$$

$$\left[by\ \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1},\ where\ a=-1\ \&\ n=2\right]$$

Q.24. Evaluate $\lim\limits_{x\to 2}\dfrac{\sqrt{x}-\sqrt{2}}{x-2}$.

Sol. $\lim\limits_{x\to 2}\dfrac{\sqrt{x}-\sqrt{2}}{x-2}=\dfrac{\sqrt{2}-\sqrt{2}}{2-2}=\dfrac{0}{0}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$$\therefore\ \lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}=\lim_{x\to 2}\frac{x^{1/2}-2^{1/2}}{x-2}=\frac{1}{2}\times(2)^{\frac{1}{2}-1}=\frac{1}{2}\times(2)^{-\frac{1}{2}}=\frac{1}{2\sqrt{2}}$$

$$\left[by\ \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1},\ where\ a=2\ \&\ n=\frac{1}{2}\right]$$

Q.25. Evaluate $\lim\limits_{x\to 5}\dfrac{x^{\frac{1}{3}}-5^{\frac{1}{3}}}{x-5}$.

Sol. $\lim\limits_{x\to 5}\dfrac{x^{\frac{1}{3}}-5^{\frac{1}{3}}}{x-5}=\dfrac{5^{\frac{1}{3}}-5^{\frac{1}{3}}}{5-5}=\dfrac{0}{0}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$$\therefore\ \lim_{x\to 5}\frac{x^{\frac{1}{3}}-5^{\frac{1}{3}}}{x-5}=\frac{1}{3}\times(5)^{\frac{1}{3}-1}=\frac{1}{3}\times(5)^{-\frac{2}{3}}=\frac{1}{3(5)^{\frac{2}{3}}}$$

$$\left[by\ \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1},\ where\ a=5\ \&\ n=\frac{1}{3}\right]$$

Q.26. Evaluate $\lim\limits_{x\to 0}(\sin x+\cos x)$.

Sol. $\lim\limits_{x\to 0}(\sin x+\cos x)=\sin 0+\cos 0=0+1=1$

Q.27. Evaluate $\lim\limits_{x\to \frac{\pi}{2}}(\sin x-\cos x)$.

Sol. $\lim_{x\to\frac{\pi}{2}}(\sin x-\cos x)=\sin\frac{\pi}{2}-\cos\frac{\pi}{2}=1-0=1$

Q.28. Evaluate $\lim_{x\to 0}(2\sin x-4\cos x+3\tan x)$.

Sol. $\lim_{x\to 0}(2\sin x-4\cos x+3\tan x)=2\sin 0-4\cos 0+3\tan 0=(2\times 0)-(4\times 1)+(3\times 0)=-4$

Q.29. Evaluate $\lim_{x\to 0}\frac{\sin 5x}{6x}$.

Sol. $\lim_{x\to 0}\frac{\sin 5x}{6x}=\frac{\sin(5\times 0)}{6\times 0}=\frac{\sin(0)}{0}=\frac{0}{0}$ $\left(\frac{0}{0}\ form\right)$

$\therefore\ \lim_{x\to 0}\frac{\sin 5x}{6x}=\lim_{x\to 0}\frac{\sin 5x}{5x}\times\frac{5}{6}=1\times\frac{5}{6}=\frac{5}{6}$ $\left[by\ \lim_{x\to 0}\frac{\sin x}{x}=1\right]$

Q.30. Evaluate $\lim_{x\to 0}\frac{4x}{\sin 2x}$.

Sol. $\lim_{x\to 0}\frac{4x}{\sin 2x}=\frac{4\times 0}{\sin(2\times 0)}=\frac{0}{\sin(0)}=\frac{0}{0}$ $\left(\frac{0}{0}\ form\right)$

$\therefore\ \lim_{x\to 0}\frac{4x}{\sin 2x}=\lim_{x\to 0}\frac{4x}{\frac{\sin 2x}{2x}\times 2x}=\lim_{x\to 0}\frac{4x}{1\times 2x}=\frac{4}{2}=2$ $\left[by\ \lim_{x\to 0}\frac{\sin x}{x}=1\right]$

Q.31. Evaluate $\lim_{x\to 0}\frac{\sin x^\circ}{x}$.

Sol. $\lim_{x\to 0}\frac{\sin x^\circ}{x}=\frac{\sin 0^\circ}{0}=\frac{0}{0}$ $\left(\frac{0}{0}\ form\right)$

$\therefore\ \lim_{x\to 0}\frac{\sin x^\circ}{x}=\lim_{x\to 0}\frac{\sin x^\circ}{x^\circ}\times\frac{x^\circ}{x}=\lim_{x\to 0}\frac{x^\circ}{x}=\lim_{x\to 0}\frac{x\times\frac{\pi}{180}}{x}=\frac{\pi}{180}$

$\left[by\ \lim_{x\to 0}\frac{\sin x}{x}=1\ and\ x^\circ=\frac{x\times\pi}{180}\right]$

Q.32. Evaluate $\lim_{x\to 0}\frac{\tan 6x}{3x}$.

Sol. $\lim_{x\to 0}\frac{\tan 6x}{3x}=\frac{\tan(6\times 0)}{3\times 0}=\frac{\tan(0)}{0}=\frac{0}{0}$ $\left(\frac{0}{0}\ form\right)$

$$\therefore \lim_{x\to 0}\frac{\tan 6x}{3x}=\lim_{x\to 0}\frac{\tan 6x}{6x}\times\frac{6}{3}=1\times\frac{6}{3}=2 \qquad \left[by \quad \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

Q.33. Evaluate $\lim_{x\to 0}\frac{\sin x^2}{5x}$.

Sol. $$\lim_{x\to 0}\frac{\sin x^2}{5x}=\frac{\sin(0^2)}{5\times 0}=\frac{\sin(0)}{0}=\frac{0}{0} \qquad \left(\frac{0}{0}\, form\right)$$

$$\therefore \lim_{x\to 0}\frac{\sin x^2}{5x}=\lim_{x\to 0}\left[\frac{\sin x^2}{x^2}\times\frac{x^2}{5x}\right]=1\times\lim_{x\to 0}\frac{x^2}{5x}=\lim_{x\to 0}\frac{x}{5}=\frac{0}{5}=0 \qquad \left[by \quad \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

Q.34. Evaluate $\lim_{x\to 0}\frac{9x}{\tan 3x}$.

Sol. $$\lim_{x\to 0}\frac{9x}{\tan 3x}=\frac{9\times 0}{\tan(3\times 0)}=\frac{0}{\tan(0)}=\frac{0}{0} \qquad \left(\frac{0}{0}\, form\right)$$

$$\therefore \lim_{x\to 0}\frac{9x}{\tan 3x}=\lim_{x\to 0}\frac{9x}{\frac{\tan 3x}{3x}\times 3x}=\lim_{x\to 0}\frac{9x}{1\times 3x}=\frac{9}{3}=3 \qquad \left[by \quad \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

Q.35. Evaluate $\lim_{x\to 0}\frac{\sin px}{\tan qx}$.

Sol. $$\lim_{x\to 0}\frac{\sin px}{\tan qx}=\frac{\sin(p\times 0)}{\tan(q\times 0)}=\frac{\sin(0)}{\tan(0)}=\frac{0}{0} \qquad \left(\frac{0}{0}\, form\right)$$

$$\therefore \lim_{x\to 0}\frac{\sin px}{\tan qx}=\lim_{x\to 0}\frac{\frac{\sin px}{px}\times px}{\frac{\tan qx}{qx}\times qx}=\lim_{x\to 0}\frac{1\times px}{1\times qx}=\lim_{x\to 0}\frac{p}{q}=\frac{p}{q}$$

$$\left[by \quad \lim_{x\to 0}\frac{\sin x}{x}=1 \;\&\; \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

Q.36. Evaluate $\lim_{x\to 0}\frac{\tan 4x}{\sin 2x}$.

Sol. $$\lim_{x\to 0}\frac{\tan 4x}{\sin 2x}=\frac{\tan(4\times 0)}{\sin(2\times 0)}=\frac{\tan(0)}{\sin(0)}=\frac{0}{0} \qquad \left(\frac{0}{0}\, form\right)$$

$$\therefore \lim_{x\to 0}\frac{\tan 4x}{\sin 2x}=\lim_{x\to 0}\frac{\frac{\tan 4x}{4x}\times 4x}{\frac{\sin 2x}{2x}\times 2x}=\lim_{x\to 0}\frac{1\times 4x}{1\times 2x}=\lim_{x\to 0}\frac{4}{2}=2$$

$$\left[by \;\lim_{x\to 0}\frac{\sin x}{x}=1 \;\&\; \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

Q.37. Evaluate $\lim_{x\to 0}\frac{\tan Ax}{\sin Bx}$.

Sol. $$\lim_{x\to 0}\frac{\tan Ax}{\sin Bx}=\frac{\tan(A\times 0)}{\sin(B\times 0)}=\frac{\tan(0)}{\sin(0)}=\frac{0}{0} \qquad \left(\frac{0}{0}\,form\right)$$

$$\therefore \lim_{x\to 0}\frac{\tan Ax}{\sin Bx}=\lim_{x\to 0}\frac{\frac{\tan Ax}{Ax}\times Ax}{\frac{\sin Bx}{Bx}\times Bx}=\lim_{x\to 0}\frac{1\times Ax}{1\times Bx}=\lim_{x\to 0}\frac{A}{B}=\frac{A}{B}$$

$$\left[by \;\lim_{x\to 0}\frac{\sin x}{x}=1 \;\&\; \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

Q.38. Evaluate $\lim_{x\to 0}\frac{\sin 8x}{x}$.

Sol. $$\lim_{x\to 0}\frac{\sin 8x}{x}=\frac{\sin(8\times 0)}{0}=\frac{\sin(0)}{0}=\frac{0}{0} \qquad \left(\frac{0}{0}\,form\right)$$

$$\therefore \lim_{x\to 0}\frac{\sin 8x}{x}=\lim_{x\to 0}\frac{\sin 8x}{8x}\times 8=1\times 8=8 \qquad \left[by \;\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

Q.39. Evaluate $\lim_{x\to 0}\frac{\sin 3x-\tan 2x}{x}$.

Sol. $$\lim_{x\to 0}\frac{\sin 3x-\tan 2x}{x}=\frac{\sin(3\times 0)-\tan(2\times 0)}{0}=\frac{0}{0} \qquad \left(\frac{0}{0}\,form\right)$$

$$\therefore \lim_{x\to 0}\frac{\sin 3x-\tan 2x}{x}=\lim_{x\to 0}\left[\frac{\sin 3x}{x}-\frac{\tan 2x}{x}\right]$$

$$=\lim_{x\to 0}\left[\frac{\sin 3x}{3x}\times 3-\frac{\tan 2x}{2x}\times 2\right]=1\times 3-1\times 2=3-2=1$$

$$\left[by \;\lim_{x\to 0}\frac{\sin x}{x}=1 \;\&\; \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

Q.40. Evaluate $\lim_{x\to 0}\frac{\tan 3x}{2x^2}$.

Sol. $\lim_{x\to 0}\frac{\tan 3x}{2x^2}=\frac{\tan(3\times 0)}{2(0)^2}=\frac{\tan(0)}{0}=\frac{0}{0}$ $\left(\frac{0}{0}\text{ form}\right)$

$$\therefore \lim_{x\to 0}\frac{\tan 3x}{2x^2}=\lim_{x\to 0}\frac{\tan 3x}{3x}\times\frac{3x}{2x^2}=1\times\lim_{x\to 0}\frac{3}{2x}=\frac{3}{2\times 0}=\frac{3}{0}=\infty \quad \left[by \ \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

Q.41. Evaluate $\lim_{x\to 0}\frac{\tan 4x+2x}{2x}$.

Sol. $\lim_{x\to 0}\frac{\tan 4x+2x}{2x}=\frac{\tan(4\times 0)+(2\times 0)}{(2\times 0)}=\frac{0+0}{0}=\frac{0}{0}$ $\left(\frac{0}{0}\text{ form}\right)$

$$\therefore \lim_{x\to 0}\frac{\tan 4x+2x}{2x}=\lim_{x\to 0}\left[\frac{\tan 4x}{2x}+\frac{2x}{2x}\right]=\lim_{x\to 0}\left[\left(\frac{\tan 4x}{4x}\times 2\right)+1\right]$$

$$=(1\times 2)+1=2+1=3 \quad \left[by \ \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

Q.42. Evaluate $\lim_{x\to 0}\frac{2\sin x-\tan 2x}{4}$.

Sol. $\lim_{x\to 0}\frac{2\sin x-\tan 2x}{4}=\frac{2\sin 0-\tan 0}{4}=\frac{(2\times 0)-0}{4}=\frac{0}{4}=0$

Q.43. Evaluate $\lim_{x\to 0}\frac{3x+\sin 5x}{\sin 6x+2x}$.

Sol. $\lim_{x\to 0}\frac{3x+\sin 5x}{\sin 6x+2x}=\frac{3(0)+\sin 0}{\sin 0+2(0)}=\frac{0+0}{0+0}=\frac{0}{0}$ $\left(\frac{0}{0}\text{ form}\right)$

$$\therefore \lim_{x\to 0}\frac{3x+\sin 5x}{\sin 6x+2x}=\lim_{x\to 0}\frac{\left(\frac{3x+\sin 5x}{x}\right)}{\left(\frac{\sin 6x+2x}{x}\right)}$$

(Divided numerator & denominator by 'x')

$$=\lim_{x\to 0}\frac{\frac{3x}{x}+\frac{\sin 5x}{x}}{\frac{\sin 6x}{x}+\frac{2x}{x}}=\lim_{x\to 0}\frac{3+\left(\frac{\sin 5x}{5x}\times 5\right)}{\left(\frac{\sin 6x}{6x}\times 6\right)+2}=\frac{3+(1\times 5)}{(1\times 6)+2}=\frac{8}{8}=1$$

Q.44. Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 4x - x}{6x-\sin 3x}$.

Sol. $\lim\limits_{x\to 0}\dfrac{\sin 4x - x}{6x-\sin 3x}=\dfrac{\sin 0-0}{0-\sin 0}=\dfrac{0-0}{0-0}=\dfrac{0}{0}$ $\qquad\left(\dfrac{0}{0}\ form\right)$

$$\therefore\ \lim_{x\to 0}\frac{\sin 4x - x}{6x-\sin 3x}=\lim_{x\to 0}\frac{\left(\dfrac{\sin 4x - x}{x}\right)}{\left(\dfrac{6x-\sin 3x}{x}\right)}$$

(*Divided numerator & denominator by 'x'*)

$$=\lim_{x\to 0}\frac{\dfrac{\sin 4x}{x}-\dfrac{x}{x}}{\dfrac{6x}{x}-\dfrac{\sin 3x}{x}}=\lim_{x\to 0}\frac{\left(\dfrac{\sin 4x}{4x}\times 4\right)-1}{6-\left(\dfrac{\sin 3x}{3x}\times 3\right)}=\frac{4-1}{6-3}=\frac{3}{3}=1$$

<u>Questions based on</u> $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log_e a$ **:**

Q.45. Evaluate $\lim\limits_{x\to 0}\dfrac{2^x-1}{x}$.

Sol. $\lim\limits_{x\to 0}\dfrac{2^x-1}{x}=\dfrac{2^0-1}{0}=\dfrac{1-1}{0}=\dfrac{0}{0}$ $\qquad\left(\dfrac{0}{0}\ form\right)$

$\therefore\ \lim\limits_{x\to 0}\dfrac{2^x-1}{x}=\log_e 2$ $\qquad(here\ \ a=2)$

Q.46. Evaluate $\lim\limits_{x\to 0}\dfrac{5^x-1}{x}$.

Sol. $\lim\limits_{x\to 0}\dfrac{5^x-1}{x}=\dfrac{5^0-1}{0}=\dfrac{1-1}{0}=\dfrac{0}{0}$ $\qquad\left(\dfrac{0}{0}\ form\right)$

$\therefore\ \lim\limits_{x\to 0}\dfrac{5^x-1}{x}=\log_e 5$ $\qquad(here\ \ a=5)$

Q.47. Evaluate $\lim\limits_{x\to 0}\dfrac{7^x-1}{x}$.

Sol. $\lim_{x \to 0} \frac{7^x - 1}{x} = \frac{7^0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore \lim_{x \to 0} \frac{7^x - 1}{x} = \log_e 7$ $(\text{here } a = 7)$

Q.48. Evaluate $\lim_{x \to 0} \frac{3^x - 2^x}{x}$.

Sol. $\lim_{x \to 0} \frac{3^x - 2^x}{x} = \frac{3^0 - 2^0}{0} = \frac{1-1}{0} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore \lim_{x \to 0} \frac{3^x - 2^x}{x} = \lim_{x \to 0} \frac{3^x - 1 + 1 - 2^x}{x} = \lim_{x \to 0} \frac{(3^x - 1) - (2^x - 1)}{x}$

$= \lim_{x \to 0} \left[\left(\frac{3^x - 1}{x} \right) - \left(\frac{2^x - 1}{x} \right) \right]$

$= \log_e 3 - \log_e 2 = \log_e \frac{3}{2}$ $\left(\because \log_e a - \log_e b = \log_e \frac{a}{b}\right)$

Q.49. Evaluate $\lim_{x \to 0} \frac{5^x - 7^x}{x}$.

Sol. $\lim_{x \to 0} \frac{5^x - 7^x}{x} = \frac{5^0 - 7^0}{0} = \frac{1-1}{0} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore \lim_{x \to 0} \frac{5^x - 7^x}{x} = \lim_{x \to 0} \frac{5^x - 1 + 1 - 7^x}{x} = \lim_{x \to 0} \frac{(5^x - 1) - (7^x - 1)}{x}$

$= \lim_{x \to 0} \left[\left(\frac{5^x - 1}{x} \right) - \left(\frac{7^x - 1}{x} \right) \right]$

$= \log_e 5 - \log_e 7 = \log_e \frac{5}{7}$ $\left(\because \log_e a - \log_e b = \log_e \frac{a}{b}\right)$

Q.50. Evaluate $\lim_{x \to 0} \frac{2^x - 5^x}{x}$.

Sol. $\lim_{x \to 0} \frac{2^x - 5^x}{x} = \frac{2^0 - 5^0}{0} = \frac{1-1}{0} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore \lim_{x \to 0} \frac{2^x - 5^x}{x} = \lim_{x \to 0} \frac{2^x - 1 + 1 - 5^x}{x} = \lim_{x \to 0} \frac{(2^x - 1) - (5^x - 1)}{x}$

$$= \lim_{x \to 0}\left[\left(\frac{2^x - 1}{x}\right) - \left(\frac{5^x - 1}{x}\right)\right]$$

$$= \log_e 2 - \log_e 5 = \log_e \frac{2}{5} \qquad \left(\because \log_e a - \log_e b = \log_e \frac{a}{b}\right)$$

Q.51. Evaluate $\lim_{x \to 0} \frac{4^x - 3^x}{\sin x}$.

Sol. $\lim_{x \to 0} \frac{4^x - 3^x}{\sin x} = \frac{4^0 - 3^0}{\sin 0} = \frac{1-1}{0} = \frac{0}{0}$ $\qquad \left(\frac{0}{0}\ form\right)$

$$\therefore \lim_{x \to 0} \frac{4^x - 3^x}{\sin x} = \lim_{x \to 0} \frac{4^x - 1 + 1 - 3^x}{\frac{\sin x}{x} \times x} = \lim_{x \to 0} \frac{(4^x - 1) - (3^x - 1)}{1 \times x}$$

$$= \lim_{x \to 0}\left[\left(\frac{4^x - 1}{x}\right) - \left(\frac{3^x - 1}{x}\right)\right]$$

$$= \log_e 4 - \log_e 3 = \log_e \frac{4}{3} \qquad \left(\because \log_e a - \log_e b = \log_e \frac{a}{b}\right)$$

Q.52. Evaluate $\lim_{x \to 0} \frac{5^x - 3^x}{\tan x}$.

Sol. $\lim_{x \to 0} \frac{5^x - 3^x}{\tan x} = \frac{5^0 - 3^0}{\tan 0} = \frac{1-1}{0} = \frac{0}{0}$ $\qquad \left(\frac{0}{0}\ form\right)$

$$\therefore \lim_{x \to 0} \frac{5^x - 3^x}{\tan x} = \lim_{x \to 0} \frac{5^x - 1 + 1 - 3^x}{\frac{\tan x}{x} \times x} = \lim_{x \to 0} \frac{(5^x - 1) - (3^x - 1)}{1 \times x}$$

$$= \lim_{x \to 0}\left[\left(\frac{5^x - 1}{x}\right) - \left(\frac{3^x - 1}{x}\right)\right]$$

$$= \log_e 5 - \log_e 3 = \log_e \frac{5}{3} \qquad \left(\because \log_e a - \log_e b = \log_e \frac{a}{b}\right)$$

Q.53. Evaluate $\lim_{x \to 0} \frac{p^x - q^x}{x}$.

Sol. $\lim_{x\to 0}\frac{p^x-q^x}{x}=\frac{p^0-q^0}{0}=\frac{1-1}{0}=\frac{0}{0}$ $\left(\frac{0}{0}\,form\right)$

$$\therefore \lim_{x\to 0}\frac{p^x-q^x}{x}=\lim_{x\to 0}\frac{p^x-1+1-q^x}{x}=\lim_{x\to 0}\frac{(p^x-1)-(q^x-1)}{x}$$

$$=\lim_{x\to 0}\left[\left(\frac{p^x-1}{x}\right)-\left(\frac{q^x-1}{x}\right)\right]$$

$$=\log_e p-\log_e q=\log_e\frac{p}{q} \qquad \left(\because \log_e a-\log_e b=\log_e\frac{a}{b}\right)$$

Q.54. Evaluate $\lim_{x\to 0}\frac{a^x-1}{b^x-1}$.

Sol. $\lim_{x\to 0}\frac{a^x-1}{b^x-1}=\frac{a^0-1}{b^0-1}=\frac{1-1}{1-1}=\frac{0}{0}$ $\left(\frac{0}{0}\,form\right)$

$$\therefore \lim_{x\to 0}\frac{a^x-1}{b^x-1}=\lim_{x\to 0}\frac{\left(\frac{a^x-1}{x}\right)}{\left(\frac{b^x-1}{x}\right)}$$

$(Divided\ numerator\ \&\ denominator\ by\ 'x')$

$$=\frac{\log_e a}{\log_e b}$$

Q.55. Evaluate $\lim_{x\to 0}\frac{2^x-1}{3^x-1}$.

Sol. $\lim_{x\to 0}\frac{2^x-1}{3^x-1}=\frac{2^0-1}{3^0-1}=\frac{1-1}{1-1}=\frac{0}{0}$ $\left(\frac{0}{0}\,form\right)$

$$\therefore \lim_{x\to 0}\frac{2^x-1}{3^x-1}=\lim_{x\to 0}\frac{\left(\frac{2^x-1}{x}\right)}{\left(\frac{3^x-1}{x}\right)}$$

$(Divided\ numerator\ \&\ denominator\ by\ 'x')$

$$=\frac{\log_e 2}{\log_e 3}$$

Q.56. Evaluate $\lim_{x\to 0}\frac{7^x-1}{5^x-1}$.

Sol. $\lim_{x\to 0}\frac{7^x-1}{5^x-1}=\frac{7^0-1}{5^0-1}=\frac{1-1}{1-1}=\frac{0}{0}$ $\left(\frac{0}{0}\ form\right)$

$$\therefore\ \lim_{x\to 0}\frac{7^x-1}{5^x-1}=\lim_{x\to 0}\frac{\left(\frac{7^x-1}{x}\right)}{\left(\frac{5^x-1}{x}\right)}$$

(Divided numerator & denominator by 'x')

$$=\frac{\log_e 7}{\log_e 5}$$

Q.57. Evaluate $\lim_{x\to 0}\frac{2^x-1}{\sin x}$.

Sol. $\lim_{x\to 0}\frac{2^x-1}{\sin x}=\frac{2^0-1}{\sin 0}=\frac{1-1}{0}=\frac{0}{0}$ $\left(\frac{0}{0}\ form\right)$

$$\therefore\ \lim_{x\to 0}\frac{2^x-1}{\sin x}=\lim_{x\to 0}\frac{\left(\frac{2^x-1}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$

(Divided numerator & denominator by 'x')

$$=\frac{\log_e 2}{1}=\log_e 2$$

Q.58. Evaluate $\lim_{x\to 0}\frac{a^{\tan x}-1}{\tan x}$.

Sol. $\lim_{x\to 0}\frac{a^{\tan x}-1}{\tan x}=\frac{a^{\tan 0}-1}{\tan 0}=\frac{a^0-1}{0}=\frac{1-1}{0}=\frac{0}{0}$ $\left(\frac{0}{0}\ form\right)$

$$\therefore\ \lim_{x\to 0}\frac{a^{\tan x}-1}{\tan x}=\log_e a$$

(Because tanx → 0 as x → 0)

Q.59. Evaluate $\lim_{x\to 0}\frac{e^{\sin x}-1}{x}$.

Sol. $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-1}{x}=\dfrac{e^{\sin 0}-1}{0}=\dfrac{e^{0}-1}{0}=\dfrac{1-1}{0}=\dfrac{0}{0}$ $\left(\dfrac{0}{0}\,form\right)$

$$\therefore\ \lim_{x\to 0}\frac{e^{\sin x}-1}{x}=\lim_{x\to 0}\left[\frac{e^{\sin x}-1}{\sin x}\times\frac{\sin x}{x}\right]$$

$$=\log_e e\times 1=1\times 1=1 \qquad (\because \log_e e=1)$$

Q.60. Evaluate $\lim\limits_{x\to 0}\dfrac{e^{x^2}-1}{x\tan x}$.

Sol. $\lim\limits_{x\to 0}\dfrac{e^{x^2}-1}{x\tan x}=\dfrac{e^{(0)^2}-1}{0\times\tan 0}=\dfrac{e^{0}-1}{0}=\dfrac{1-1}{0}=\dfrac{0}{0}$ $\left(\dfrac{0}{0}\,form\right)$

$$\therefore\ \lim_{x\to 0}\frac{e^{x^2}-1}{x\tan x}=\lim_{x\to 0}\left[\frac{e^{x^2}-1}{x^2}\times\frac{x^2}{x\tan x}\right]$$

$$=\log_e e\times\lim_{x\to 0}\frac{x^2}{x\tan x}=1\times\lim_{x\to 0}\frac{x}{\tan x}=1 \qquad (\because \log_e e=1)$$

1.2 Differentiation by First Principle

Increment: Increment is the quantity by which the value of variable changes. It may be positive or negative. e.g. suppose the value of a variable x changes from 5 to 5.3 then 0.3 is the increment in x. Similarly, if the value of variable x changes from 5 to 4.5 then -0.5 is the increment in x.

Usually δx represents the increment in x, δy represents the increment in y, δz represents the increment in z etc.

Derivative or Differential Co-efficient: If y is a function of x. Let δx be the increment in x and δy be the corresponding increment in y, then $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ $(if\ it\ exists)$ is called the derivative or differential co-efficient of y with respect to x and is dented by $\frac{dy}{dx}$.

i.e. $$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

First Principle Method of Differentiation:

Let $y = f(x)$ (1)

Let δx be the increment in x and δy be the corresponding increment in y, then

$$y + \delta y = f(x + \delta x) \qquad (2)$$

Subtracting equation (1) from equation (2), we get

$$y + \delta y - y = f(x + \delta x) - f(x)$$

$$\Rightarrow \qquad \delta y = f(x + \delta x) - f(x)$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Taking limit $\delta x \to 0$ on both sides, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

If this limit exists, we write it as

$$\frac{dy}{dx} = f'(x)$$

where $f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$.

This is called the differentiation or derivative of the function $f(x)$ with respect to x.

Notations: The first order derivative of the function $f(x)$ with respect to x can be represented in the following ways:

$$\frac{d}{dx}(f(x)),\ \frac{df}{dx},\ f'(x),\ f_1(x)\ etc.$$

Similarly, the first order derivative of y with respect to x can be represented as:

$$\frac{dy}{dx},\ y',\ y_1\ etc.$$

Physical Interpretation of Derivatives:

Let the variable t represents the time and the function $f(t)$ represents the distance travelled in time t.

We know that $Speed = \frac{Distance\ travelled}{Time\ taken}$

If time interval is between $'a'$ & $'a+h'$. Here h be increment in a. Then the speed in that interval is given by

$$\frac{Distance\ travelled\ upto\ time\ (a+h) - Distance\ travelled\ upto\ time\ (a)}{leng\ \ of\ time\ interval}$$

$$= \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}$$

If we take $h \to 0$ then $\frac{f(a+h) - f(a)}{h}$ approaches the speed at time $t = a$. Thus we can say that derivative is related in the similar way as speed is related to the distance travelled by a moving particle.

Q.1. Differentiate x^n with respect to x by First Principle Method.

Ans. Let $y = x^n$ (1)

Let δx be the increment in x and δy be the corresponding increment in y, then

$$y+\delta y=(x+\delta x)^n \qquad (2)$$

Subtracting equation (1) from equation (2), we get

$$y+\delta y-y=(x+\delta x)^n-x^n$$

$$\Rightarrow \qquad \delta y=\left[x^n+n x^{n-1}\,\delta x+\frac{n(n-1)x^{n-2}(\delta x)^2}{2!}+\ldots\right]-x^n$$

$$\Rightarrow \qquad \delta y=n x^{n-1}\,\delta x+\frac{n(n-1)x^{n-2}(\delta x)^2}{2!}+\ldots$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x}=\frac{n x^{n-1}\,\delta x+\frac{n(n-1)x^{n-2}(\delta x)^2}{2!}+\ldots}{\delta x}=n x^{n-1}+\frac{n(n-1)x^{n-2}\,\delta x}{2!}+\ldots$$

Taking limit $\delta x\to 0$ on both sides, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[n x^{n-1}+\frac{n(n-1)x^{n-2}\,\delta x}{2!}+\ldots\right]$$

$$\Rightarrow \qquad \frac{dy}{dx}=n x^{n-1}$$

Hence $\quad \frac{d}{dx}\left(x^n\right)=n x^{n-1}$.

Q.2. Differentiate $\sin x$ with respect to x by First Principle Method.

Ans. Let $y=\sin x$ (1)

Let δx be the increment in x and δy be the corresponding increment in y, then

$$y+\delta y=\sin(x+\delta x) \qquad (2)$$

Subtracting equation (1) from equation (2), we get

$$y+\delta y-y=\sin(x+\delta x)-\sin x$$

$$\Rightarrow \qquad \delta y=2\cos\left(\frac{x+\delta x+x}{2}\right)\sin\left(\frac{x+\delta x-x}{2}\right)$$

$$\Rightarrow \qquad \delta y = 2\cos\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{2\cos\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

Taking limit $\delta x \to 0$ on both sides, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[\frac{2\cos\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}\right]$$

$$\Rightarrow \qquad \frac{dy}{dx} = \lim_{\delta x\to 0}\left[\frac{2\cos\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{2\left(\frac{\delta x}{2}\right)}\right]$$

$$\Rightarrow \qquad \frac{dy}{dx} = \lim_{\delta x\to 0}\left[\cos\left(\frac{2x+\delta x}{2}\right)\left(\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}\right)\right]$$

$$\Rightarrow \qquad \frac{dy}{dx} = \lim_{\delta x\to 0}\left[\cos\left(\frac{2x+\delta x}{2}\right)\times 1\right]$$

$$\Rightarrow \qquad \frac{dy}{dx} = \cos\left(\frac{2x+0}{2}\right) = \cos x$$

Hence $\frac{d}{dx}(\sin x) = \cos x$.

Q.3. Differentiate $\cos x$ with respect to x by First Principle Method.

Ans. Let $y = \cos x$ (1)

Let δx be the increment in x and δy be the corresponding increment in y, then

$$y + \delta y = \cos(x + \delta x) \qquad (2)$$

Subtracting equation (1) from equation (2), we get

$$y+\delta y-y=\cos(x+\delta x)-\cos x$$

$$\Rightarrow \qquad \delta y=-2\sin\left(\frac{x+\delta x+x}{2}\right)\sin\left(\frac{x+\delta x-x}{2}\right)$$

$$\Rightarrow \qquad \delta y=-2\sin\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x}=\frac{-2\sin\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

Taking limit $\delta x\to 0$ on both sides, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[\frac{-2\sin\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}\right]$$

$$\Rightarrow \qquad \frac{dy}{dx}=\lim_{\delta x\to 0}\left[\frac{-2\sin\left(\frac{2x+\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{2\left(\frac{\delta x}{2}\right)}\right]$$

$$\Rightarrow \qquad \frac{dy}{dx}=\lim_{\delta x\to 0}\left[-\sin\left(\frac{2x+\delta x}{2}\right)\left(\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}\right)\right]$$

$$\Rightarrow \qquad \frac{dy}{dx}=\lim_{\delta x\to 0}\left[-\sin\left(\frac{2x+\delta x}{2}\right)\times 1\right]$$

$$\Rightarrow \qquad \frac{dy}{dx}=-\sin\left(\frac{2x+0}{2}\right)=-\sin x$$

Hence $\frac{d}{dx}(\cos x)=-\sin x$.

Q.4. Differentiate e^x with respect to x by First Principle Method.

Ans. Let $y = e^x$ (1)

Let δx be the increment in x and δy be the corresponding increment in y, then

$$y + \delta y = e^{x+\delta x} \quad (2)$$

Subtracting equation (1) from equation (2), we get

$$y + \delta y - y = e^{x+\delta x} - e^x$$

$$\Rightarrow \quad \delta y = e^x \left(e^{\delta x} - 1\right)$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{e^x \left(e^{\delta x} - 1\right)}{\delta x}$$

Taking limit $\delta x \to 0$ on both sides, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left[\frac{e^x \left(e^{\delta x} - 1\right)}{\delta x}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x \times \log_e e = e^x \times 1 = e^x$$

Hence $\frac{d}{dx}\left(e^x\right) = e^x$.

1.3 Differentiation of Sum, Product and Quotient

AND

2.1 Differentiation of trigonometric functions, inverse trigonometric functions, Logarithmic differentiation, successive differentiation (upto 2nd order)

Basic Properties of Differentiation:

If $f(x)$ are $g(x)$ differentiable functions, then

(i) $\frac{d}{dx}(K) = 0$ where K is some constant.

(ii) $\frac{d}{dx}(K.f(x)) = K.\frac{d}{dx}(f(x))$ where K is some constant.

(iii) $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

(iv) $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$

(v) $\frac{d}{dx}[f(x).g(x)] = f(x).\frac{d}{dx}(g(x)) + g(x).\frac{d}{dx}(f(x))$

This property is known as Product Rule of differentiation.

(vi) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$ provided that $g(x) \neq 0$

This property is known as Quotient Rule of differentiation.

Some Basic Formulas of Differentiation:

(i) $\frac{d}{dx}(x^n) = n x^{n-1}$ this is known as power formula, here n is any real number.

(ii) $\frac{d}{dx}(a^x) = a^x \log_e a \qquad here\ a>0\ \&\ a \neq 1$

(iii) $\frac{d}{dx}(e^x) = e^x \log_e e = e^x$

(iv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$

Q.1. Differentiate $y = x^{10}$ with respect to x.

Sol. Given that $y = x^{10}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^{10}) = 10x^9$$

Q.2. Differentiate $y = \sqrt{x}$ with respect to x.

Sol. Given that $y = \sqrt{x}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}x^{\frac{1}{2}-1}$$

$$=\frac{1}{2}x^{\frac{1-2}{2}}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2x^{\frac{1}{2}}}=\frac{1}{2\sqrt{x}}$$

Q.3. Differentiate $y=x^{\frac{1}{3}}$ with respect to x.

Sol. Given that $y=x^{\frac{1}{3}}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{\frac{1}{3}}\right)=\frac{1}{3}x^{\frac{1}{3}-1}$$

$$=\frac{1}{3}x^{\frac{1-3}{3}}=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3x^{\frac{2}{3}}}$$

Q.4. Differentiate $y=\frac{1}{\sqrt{x}}$ with respect to x.

Sol. Given that $y=\frac{1}{\sqrt{x}}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)=\frac{d}{dx}\left(\frac{1}{x^{\frac{1}{2}}}\right)=\frac{d}{dx}\left(x^{-\frac{1}{2}}\right)$$

$$=-\frac{1}{2}x^{-\frac{1}{2}-1}=-\frac{1}{2}x^{\frac{-1-2}{2}}=-\frac{1}{2}x^{-\frac{3}{2}}=-\frac{1}{2x^{\frac{3}{2}}}$$

Q.5. Differentiate $y=5-x^6$ with respect to x.

Sol. Given that $y=5-x^6$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(5-x^6\right)=\frac{d}{dx}(5)-\frac{d}{dx}\left(x^6\right)$$

$$=0-6x^5=-6x^5$$

Q.6. Differentiate $y=x^{-\frac{5}{2}}$ with respect to x.

Sol. Given that $y=x^{-\frac{5}{2}}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{-\frac{5}{2}}\right)=-\frac{5}{2}x^{-\frac{5}{2}-1}$$

$$=-\frac{5}{2}x^{\frac{-5-2}{2}}=-\frac{5}{2}x^{-\frac{7}{2}}=-\frac{5}{2x^{\frac{7}{2}}}$$

Q.7. Differentiate $y=\sqrt{x}-\frac{1}{\sqrt{x}}$ with respect to x.

Sol. Given that $y=\sqrt{x}-\frac{1}{\sqrt{x}}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)=\frac{d}{dx}\left(\sqrt{x}\right)-\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)$$

$$=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)-\frac{d}{dx}\left(x^{-\frac{1}{2}}\right)=\frac{1}{2}x^{\frac{1}{2}-1}-\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$$

$$=\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}$$

Q.8. Differentiate $y=2-x+3x^2$ with respect to x.

Sol. Given that $y=2-x+3x^2$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(2-x+3x^2\right)=\frac{d}{dx}(2)-\frac{d}{dx}(x)+\frac{d}{dx}\left(3x^2\right)$$

$$= 0-1+3(2x)=-1+6x$$

Q.9. Differentiate $y=(x+3)^2$ with respect to x.

Sol. Given that $y=(x+3)^2 = x^2+9+6x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2+9+6x\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(9)+\frac{d}{dx}(6x)$$

$$=2x+0+6=2x+6$$

Q.10. Differentiate $y=(x+3)(x-1)$ with respect to x.

Sol. Given that $y=(x+3)(x-1)= x^2-x+3x-3=x^2+2x-3$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2+2x-3\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(2x)-\frac{d}{dx}(3)$$

$$=2x+2(1)-0=2x+2$$

Q.11. Differentiate $y=e^x.a^x+2x^3-\log x$ with respect to x.

Sol. Given that $y=e^x.a^x+2x^3-\log x=(ea)^x+2x^3-\log x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left((ea)^x+2x^3-\log x\right)=\frac{d}{dx}(ea)^x+2\frac{d}{dx}\left(x^3\right)-\frac{d}{dx}(\log x)$$

$$= (ea)^x\log_e(ea)+2\left(3x^2\right)-\frac{1}{x}= (ea)^x\log_e(ea)+6x^2-\frac{1}{x}$$

[Note that if base of log is not given then it is supposed to be log with base 'e']

Q.12. Differentiate $y=\left(3t^2-9\right)2^t$ with respect to t.

Sol. Given that $y=\left(3t^2-9\right)2^t$

Differentiating it with respect to t, we get

$$\frac{dy}{dt}=\frac{d}{dt}\left(\left(3t^2-9\right)2^t\right)=\left(3t^2-9\right)\frac{d}{dt}\left(2^t\right)+2^t.\frac{d}{dt}\left(3t^2-9\right)$$

$$=(3t^2-9)2^t.\log_e 2+2^t.(3\times 2t-0)$$

$$=(3t^2-9)2^t.\log_e 2+2^t.6t$$

$$=2^t\left\{(3t^2-9)\log_e 2+6t\right\}$$

Q.13. Differentiate $y=\dfrac{x^2+7}{x}$ with respect to x.

Sol. Given that $y=\dfrac{x^2+7}{x}=\dfrac{x^2}{x}+\dfrac{7}{x}=x+7x^{-1}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x+7x^{-1})=\frac{dx}{dx}+7\frac{d}{dx}(x^{-1})$$

$$=1+7(-1)x^{-1-1}=1-7x^{-2}$$

<u>Some Basic Formulas of Differentiation of Trigonometric and Inverse Trigonometric Functions</u> :

(i) $\dfrac{d}{dx}(\sin x)=\cos x$

(ii) $\dfrac{d}{dx}(\cos x)=-\sin x$

(iii) $\dfrac{d}{dx}(\tan x)=\sec^2 x$

(iv) $\dfrac{d}{dx}(\sec x)=\sec x\tan x$

(v) $\dfrac{d}{dx}(\cot x)=-\cos ec^2 x$

(vi) $\dfrac{d}{dx}(\cos ecx)=-\cos ecx\cot x$

(vii) $\dfrac{d}{dx}(\sin^{-1} x)=\dfrac{1}{\sqrt{1-x^2}}$

(viii) $\dfrac{d}{dx}(\cos^{-1} x)=\dfrac{-1}{\sqrt{1-x^2}}$

(ix) $\dfrac{d}{dx}(\tan^{-1} x)=\dfrac{1}{1+x^2}$

(x) $\dfrac{d}{dx}(\cot^{-1} x)=\dfrac{-1}{1+x^2}$

(xi) $\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}}$

(xii) $\frac{d}{dx}\left(\operatorname{cosec}^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}$

Q.16. Differentiate $y=\sin x-e^x+2^x$ with respect to x.

Sol. Given that $y=\sin x-e^x+2^x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sin x-e^x+2^x\right)=\frac{d}{dx}(\sin x)-\frac{d}{dx}\left(e^x\right)+\frac{d}{dx}\left(2^x\right)$$

$$=\cos x-e^x+2^x\log_e 2$$

Q.17. Differentiate $y=2\log x-5\sec x$ with respect to x.

Sol. Given that $y=2\log x-5\sec x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(2\log x-5\sec x\right)=2\frac{d}{dx}(\log x)-5\frac{d}{dx}(\sec x)$$

$$=\frac{2}{x}-5\sec x\tan x$$

Q.18. Differentiate $y=5\sin^{-1}x$ with respect to x.

Sol. Given that $y=5\sin^{-1}x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(5\sin^{-1}x\right)=5\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{5}{\sqrt{1-x^2}}$$

Q.19. Differentiate $y=\cos x-\tan^{-1}x$ with respect to x.

Sol. Given that $y=\cos x-\tan^{-1}x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cos x-\tan^{-1}x\right)=\frac{d}{dx}(\cos x)-\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$=-\sin x-\frac{1}{1+x^2}$$

<u>Questions based on Product Rule</u>:

Q.31. Differentiate $y=x\cos x$ with respect to x.

Sol. Given that $y=x\cos x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x\cos x)=x.\frac{d}{dx}(\cos x)+\cos x\frac{dx}{dx}$$

$$=x.(-\sin x)+\cos x.1=-x\sin x+\cos x$$

Q.32. Differentiate $y=x^2\sin x$ with respect to x.

Sol. Given that $y=x^2\sin x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2\sin x\right)=x^2\frac{d}{dx}(\sin x)+\sin x.\frac{d}{dx}\left(x^2\right)$$

$$=x^2\cos x+\sin x.2x=x^2\cos x+2x\sin x$$

Q.33. Differentiate $y=\sin x\cos x$ with respect to x.

Sol. Given that $y=\sin x\cos x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\sin x\cos x)=\sin x.\frac{d}{dx}(\cos x)+\cos x\frac{d}{dx}(\sin x)$$

$$=\sin x.(-\sin x)+\cos x.(\cos x)=-\sin^2 x+\cos^2 x$$

Q.34. Differentiate $y=\cos x\log x$ with respect to x.

Sol. Given that $y=\cos x\log x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\cos x\log x)=\cos x.\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(\cos x)$$

$$=\cos x\times\frac{1}{x}+\log x\times(-\sin x)=\frac{\cos x}{x}-\log x.\sin x$$

Q.35. Differentiate $y=\log x\tan x$ with respect to x.

Sol. Given that $y=\log x\tan x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\log x\tan x)=\log x.\frac{d}{dx}(\tan x)+\tan x\frac{d}{dx}(\log x)$$

$$=\log x\times\left(\sec^2 x\right)+\tan x\times\frac{1}{x}=\log x.\sec^2 x+\frac{\tan x}{x}$$

Q.36. Differentiate $y=(2x+5)\log x$ with respect to x.

Sol. Given that $y=(2x+5)\log x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}((2x+5)\log x)=(2x+5).\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(2x+5)$$

$$=(2x+5)\times\frac{1}{x}+\log x\times 2=\frac{2x+5}{x}+2\log x$$

Q.38. Differentiate $y=\left(t^3+8\right)\tan^{-1}t$ with respect to t.

Sol. Given that $y=\left(t^3+8\right)\tan^{-1}t$

Differentiating it with respect to t, we get

$$\frac{dy}{dt}=\frac{d}{dt}\left(\left(t^3+8\right)\tan^{-1}t\right)=\left(t^3+8\right)\frac{d}{dt}\left(\tan^{-1}t\right)+\tan^{-1}t.\frac{d}{dt}\left(t^3+8\right)$$

$$=\left(t^3+8\right)\left(\frac{1}{1+t^2}\right)+\tan^{-1}t.\left(3t^2+0\right)$$

$$=\frac{t^3+8}{1+t^2}+3t^2\tan^{-1}t$$

<u>Questions based on Quotient Rule</u>:

Q.40. Differentiate $y = \frac{\sin x}{x}$ with respect to x.

Sol. Given that $y = \frac{\sin x}{x}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x.\frac{d}{dx}(\sin x) - \sin x.\frac{dx}{dx}}{x^2}$$

$$= \frac{x.\cos x - \sin x.1}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

Q.41. Differentiate $y = \frac{\log x}{\tan x}$ with respect to x.

Sol. Given that $y = \frac{\log x}{\tan x}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log x}{\tan x}\right) = \frac{\tan x.\frac{d}{dx}(\log x) - \log x.\frac{d}{dx}(\tan x)}{\tan^2 x}$$

$$= \frac{\left(\frac{\tan x}{x} - \log x.\sec^2 x\right)}{\tan^2 x} = \frac{\left(\frac{\tan x - x\log x.\sec^2 x}{x}\right)}{\tan^2 x}$$

$$= \frac{\tan x - x\log x.\sec^2 x}{x\tan^2 x}$$

Q.42. Differentiate $y = \frac{x^2+1}{\sin x}$ with respect to x.

Sol. Given that $y = \frac{x^2+1}{\sin x}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{\sin x}\right) = \frac{\sin x.\frac{d}{dx}(x^2+1) - (x^2+1).\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$=\frac{\sin x.2x-(x^2+1).\cos x}{\sin^2 x}=\frac{2x\sin x-(x^2+1)\cos x}{\sin^2 x}$$

Chain Rule: If $f(x)$ and $g(x)$ are two differentiable functions then

$$\frac{d}{dx}(f(g(x)))=f'(g(x)).\frac{d}{dx}(g(x))=f'(g(x)).g'(x)$$

So, we may generalize our basic formulas as:

(i) $\frac{d}{dx}\left((f(x))^n\right)=n(f(x))^{n-1}.f'(x)$ here n is any real number.

(ii) $\frac{d}{dx}(\sin(f(x)))=\cos(f(x)).f'(x)$ etc.

Questions based on Chain Rule:

Q.20. Differentiate $y=\sin(2x+1)$ with respect to x.

Sol. Given that $y=\sin(2x+1)$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\sin(2x+1)=\cos(2x+1).\frac{d}{dx}(2x+1)$$

$$=\cos(2x+1).(2\times1+0)=2\cos(2x+1)$$

Logarithmic Differentiation :

Let $f(x)$ and $g(x)$ are two differentiable function and $y=f(x)^{g(x)}$

To differentiate y, first we take logarithm of y:

$$\log y=\log\left(f(x)^{g(x)}\right)$$

$$\log y=g(x)\log(f(x)) \qquad \left(\log a^b=b\log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y)=\frac{d}{dx}(g(x)\log(f(x)))$$

$$\frac{1}{y}\frac{dy}{dx}=g(x)\frac{d}{dx}(\log f(x))+\log f(x)\frac{d}{dx}(g(x))$$

$$\frac{1}{y}\frac{dy}{dx}=g(x)\frac{1}{f(x)}f'(x)+\log f(x)g'(x)$$

$$\frac{dy}{dx}=y\left[\frac{g(x)}{f(x)}f'(x)+\log f(x)g'(x)\right]$$

$$\frac{dy}{dx}=f(x)^{g(x)}\left[\frac{g(x)}{f(x)}f'(x)+\log f(x)g'(x)\right]$$

Questions based on Derivative of $f(x)^{g(x)}$ or Logarithmic Differentiation :

Q.55. Differentiate $y=x^x$ with respect to x.

Sol. Given that $y=x^x$

Taking logarithm on both sides, we get

$$\log y=\log x^x$$

$$\log y=x\log x \qquad \left(\log a^b=b\log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y)=\frac{d}{dx}(x\log x)$$

$$\frac{1}{y}\frac{dy}{dx}=x\frac{d}{dx}(\log x)+\log x\frac{dx}{dx}$$

$$\frac{1}{y}\frac{dy}{dx}=x.\frac{1}{x}+\log x.1$$

$$\frac{1}{y}\frac{dy}{dx}=1+\log x$$

$$\frac{dy}{dx}=y(1+\log x)$$

$$\frac{dy}{dx} = x^x\left(1+\log x\right)$$

Q.56. Differentiate $y = x^{\sin x}$ with respect to x.

Sol. Given that $y = x^{\sin x}$

Taking logarithm on both sides, we get

$$\log y = \log x^{\sin x}$$

$$\log y = \sin x.\log x \qquad \left(\log a^b = b\log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\sin x.\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x.\frac{1}{x} + \log x.\cos x$$

$$\frac{dy}{dx} = y\left(\frac{\sin x}{x} + \log x\cos x\right)$$

$$\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x\cos x\right)$$

Q.57. Differentiate $y = (\cos x)^x$ with respect to x.

Sol. Given that $y = (\cos x)^x$

Taking logarithm on both sides, we get

$$\log y = \log(\cos x)^x$$

$$\log y = x\log(\cos x) \qquad \left(\log a^b = b\log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(x\log(\cos x))$$

$$\frac{1}{y}\frac{dy}{dx}=x\frac{d}{dx}(\log(\cos x))+\log(\cos x)\frac{dx}{dx}$$

$$\frac{1}{y}\frac{dy}{dx}=x\frac{1}{\cos x}\frac{d}{dx}(\cos x)+\log(\cos x).1$$

$$\frac{1}{y}\frac{dy}{dx}=x\frac{1}{\cos x}(-\sin x)+\log(\cos x)$$

$$\frac{dy}{dx}=y\ (-x\tan x+\log(\cos x))$$

$$\frac{dy}{dx}=(\cos x)^x\ (-x\tan x+\log(\cos x))$$

Q.58. Differentiate $y=x^{\cos x}$ with respect to x.

Sol. Given that $y=x^{\cos x}$

Taking logarithm on both sides, we get

$$\log y=\log x^{\cos x}$$

$$\log y=\cos x.\log x \qquad (\log a^b=b\log a)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y)=\frac{d}{dx}(\cos x.\log x)$$

$$\frac{1}{y}\frac{dy}{dx}=\cos x\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(\cos x)$$

$$\frac{1}{y}\frac{dy}{dx}=\cos x.\frac{1}{x}+\log x.(-\sin x)$$

$$\frac{dy}{dx}=y\left(\frac{\cos x}{x}-\log x\sin x\right)$$

$$\frac{dy}{dx}=x^{\cos x}\left(\frac{\cos x}{x}-\log x\sin x\right)$$

Q.59. Differentiate $y=(\sin x)^{\cos x}$ with respect to x.

Sol. Given that $y=(\sin x)^{\cos x}$

Taking logarithm on both sides, we get

$$\log y=\log(\sin x)^{\cos x}$$

$$\log y=\cos x\log(\sin x) \qquad \left(\log a^{b}=b\log a\right)$$

Differentiating it with respect to x, we get

$$\frac{d}{dx}(\log y)=\frac{d}{dx}(\cos x\log(\sin x))$$

$$\frac{1}{y}\frac{dy}{dx}=\cos x\frac{d}{dx}(\log(\sin x))+\log(\sin x)\frac{d}{dx}(\cos x)$$

$$\frac{1}{y}\frac{dy}{dx}=\cos x\frac{1}{\sin x}\frac{d}{dx}(\sin x)+\log(\sin x).(-\sin x)$$

$$\frac{1}{y}\frac{dy}{dx}=\cot x\cos x-\sin x\log(\sin x)$$

$$\frac{dy}{dx}=y\ (\cot x\cos x-\sin x\log(\sin x))$$

$$\frac{dy}{dx}=(\sin x)^{\cos x}(\cot x\cos x-\sin x\log(\sin x))$$

Successive Differentiation or Higher Order Derivative:

Let $y=f(x)$ be a differentiable function, then $\frac{dy}{dx}$ represents the first order derivative of y with respect to x. If we may further differentiate it i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right)$,then it is called second order derivative of y with respect to x. Some other way to represent second order derivative of y with respect to x: $\frac{d^2y}{dx^2}$, y'' , y_2 .

So, successive derivatives of y with respect to x can be represented as

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ .\ .\ .\ ,\ \frac{d^ny}{dx^n}\ etc.$$

Q.64. If $y = x^8 - 12x^5 + 5x^3 - 12$, find $\frac{d^2y}{dx^2}$.

Ans. Given that $y = x^8 - 12x^5 + 5x^3 - 12$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^8 - 12x^5 + 5x^3 - 12\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = 8x^7 - 60x^4 + 15x^2$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(8x^7 - 60x^4 + 15x^2\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 56x^6 - 240x^3 + 30x$$

Q.65. If $y = \log(\sin x) + e^{5x}$, find $\frac{d^2y}{dx^2}$.

Ans. Given that $y = \log(\sin x) + e^{5x}$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log(\sin x) + e^{5x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sin x}\frac{d}{dx}(\sin x) + 5e^{5x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos x}{\sin x} + 5e^{5x} = \cot x + 5e^{5x}$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\cot x + 5e^{5x}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\cos ec^2 x + 25e^{5x}$$

Q.66. If $y = x^3 . e^{-2x}$, find $\frac{d^2 y}{dx^2}$ $at\ x = 3$.

Ans. Given that $y = x^3 . e^{-2x}$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 . e^{-2x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = x^3 \frac{d}{dx}\left(e^{-2x}\right) + e^{-2x}\frac{d}{dx}\left(x^3\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = x^3\left(-2e^{-2x}\right) + e^{-2x}\left(3x^2\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = -2x^3 e^{-2x} + 3x^2 e^{-2x} = \left(-2x^3 + 3x^2\right)e^{-2x}$$

Again differentiating with respect to x, we get

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\left(-2x^3 + 3x^2\right)e^{-2x}\right)$$

$$\Rightarrow \quad \frac{d^2 y}{dx^2} = \left(-2x^3 + 3x^2\right)\frac{d}{dx}e^{-2x} + e^{-2x}\frac{d}{dx}\left(-2x^3 + 3x^2\right)$$

$$\Rightarrow \quad \frac{d^2 y}{dx^2} = \left(-2x^3 + 3x^2\right)\left(-2e^{-2x}\right) + e^{-2x}\left(-6x^2 + 6x\right)$$

$$\Rightarrow \quad \frac{d^2 y}{dx^2} = \left(4x^3 - 6x^2 - 6x^2 + 6x\right)e^{-2x}$$

$$\Rightarrow \quad \frac{d^2 y}{dx^2} = \left(4x^3 - 12x^2 + 6x\right)e^{-2x}$$

Q.67. If $y = \tan^{-1} x$, prove that $\left(1 + x^2\right)\frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} = 0$.

Ans. Given that $y = \tan^{-1} x$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1} x\right)$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{1+x^2}$

$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = 1$

Again differentiating with respect to x , we get

$$(1+x^2)\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(1+x^2) = \frac{d}{dx}(1)$$

$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

Q.68. If $y = \sin Ax + \cos Ax$, prove that $\frac{d^2y}{dx^2} + A^2 y = 0$.

Ans. Given that $y = \sin Ax + \cos Ax$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin Ax + \cos Ax)$$

$\Rightarrow \quad \frac{dy}{dx} = A\cos Ax - A\sin Ax$

Again differentiating with respect to x , we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(A\cos Ax - A\sin Ax)$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -A^2 \sin Ax - A^2 \cos Ax$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -A^2(\sin Ax + \cos Ax)$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -A^2 y$

$\Rightarrow \quad \frac{d^2y}{dx^2} + A^2 y = 0$

Q.69. If $y = \sin(m\sin^{-1} x)$, prove that $(1-x^2)y_2 - xy_1 + m^2 y = 0$.

Ans. Given that $y = \sin\left(m\sin^{-1}x\right)$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin\left(m\sin^{-1}x\right)\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \cos\left(m\sin^{-1}x\right)\frac{d}{dx}\left(m\sin^{-1}x\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \cos\left(m\sin^{-1}x\right)\frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\frac{dy}{dx} = m\cos\left(m\sin^{-1}x\right)$$

Again differentiating with respect to x , we get

$$\frac{d}{dx}\left(\sqrt{1-x^2}\,\frac{dy}{dx}\right) = \frac{d}{dx}\left(m\cos\left(m\sin^{-1}x\right)\right)$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{d}{dx}\left(\sqrt{1-x^2}\right) = -m\sin\left(m\sin^{-1}x\right)\frac{d}{dx}\left(m\sin^{-1}x\right)$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{dy}{dx}\frac{x}{\sqrt{1-x^2}} = -m\sin\left(m\sin^{-1}x\right)\frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \left(\sqrt{1-x^2}\right)^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m^2\sin\left(m\sin^{-1}x\right) \qquad \left(by\ multiplying\ \sqrt{1-x^2}\ on\ both\ sides\right)$$

$$\Rightarrow \quad \left(1-x^2\right)y_2 - x\,y_1 = -m^2\,y$$

$$\Rightarrow \quad \left(1-x^2\right)y_2 - x\,y_1 + m^2\,y = 0$$

Q.70. If $y = e^{A\cot^{-1}x}$, prove that $\left(1+x^2\right)y_2 + \left(2x + A\right)y_1 = 0$.

Ans. Given that $y = e^{A\cot^{-1}x}$

Differentiating with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{A\cot^{-1}x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{A\cot^{-1}x} \frac{d}{dx}\left(A\cot^{-1}x\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{A\cot^{-1}x}\left(\frac{-A}{1+x^2}\right)$$

$$\Rightarrow \quad \left(1+x^2\right)\frac{dy}{dx} = -Ae^{A\cot^{-1}x}$$

$$\Rightarrow \quad \left(1+x^2\right)\frac{dy}{dx} = -Ay$$

Again differentiating with respect to x , we get

$$\left(1+x^2\right)\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}\left(1+x^2\right) = \frac{d}{dx}\left(-Ay\right)$$

$$\Rightarrow \quad \left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = -A\frac{dy}{dx}$$

$$\Rightarrow \quad \left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + A\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left(1+x^2\right)\frac{d^2y}{dx^2} + \left(2x+A\right)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left(1+x^2\right)y_2 + \left(2x+A\right)y_1 = 0$$

2.2 Applications of Differential Calculus

(a) Derivative as a Rate Measure:

Let y be a function of x, then $\frac{dy}{dx}$ represents the rate of change of y with respect to x.

If $\frac{dy}{dx} > 0$ then rate of change of y increases when x changes and if $\frac{dy}{dx} < 0$ then rate of change of y decreases when x changes.

Some Important Points to Remember:

(i) Usually **t** , **s** , **v** and **a** are used to represent time, displacement, velocity and acceleration respectively.

Also $v = \frac{ds}{dt}$

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = v.\frac{dv}{ds}$$

(ii) If the particle moves in the direction of *s* increasing, then $v = \frac{ds}{dt} > 0$ and if the particle moves in the direction of *s* decreasing, then $v = \frac{ds}{dt} < 0$.

(iii) If $a = 0$ then the particle is said to be moving with constant velocity and if $a < 0$ then the particle is said to have retardation.

(iv) If $\frac{dy}{dx} = 0$ then y is constant.

(v) If $y = f(x)$ be a curve then $\frac{dy}{dx}$ is said to be the slope of the curve. It is also represented by m i.e. $slope = m = \frac{dy}{dx}$.

(vi) If r be the radius, A be the area and C be the circumference of the circle then $A = \pi r^2$ & $C = 2\pi r$.

(vii) If r be the radius, S be the surface area and V be the volume of the sphere then $S = 4\pi r^2$ & $V = \frac{4}{3}\pi r^3$.

(viii) If r be the radius of base, h be the height, l be the slant length, S be the surface area and V be the volume of the cone then $S = \pi r l + \pi r^2$ & $V = \frac{1}{3}\pi r^2 h$.

(ix) If a be the length of side, S be the surface area and V be the volume of the cube then $S = 6a^2$ & $V = a^3$.

<u>Questions Related to Rate Measure</u>:

Q.1. If $y = x^3 + 5x^2 - 6x + 7$ and x increases at the rate of 3 units per minute, how fast is the slope of the curve changes when $x = 2$.

Sol. Let t represents the time.

Given that $y = x^3 + 5x^2 - 6x + 7$ (1.1)

and $\frac{dx}{dt} = 3$ (1.2)

Let m be the slope of the curve.

$\therefore \quad m = \frac{dy}{dx}$

$\Rightarrow \quad m = \frac{d}{dx}\left(x^3 + 5x^2 - 6x + 7\right) \qquad (used\ (1.1))$

$\Rightarrow \quad m = 3x^2 + 10x - 6$

Differentiating it with respect to t, we get

$$\frac{dm}{dt} = \frac{d}{dt}\left(3x^2 + 10x - 6\right)$$

$\Rightarrow \quad \frac{dm}{dt} = (6x + 10)\frac{dx}{dt}$

$\Rightarrow \quad \frac{dm}{dt} = (6x + 10).3 \qquad (used\ (1.2))$

$\Rightarrow \quad \frac{dm}{dt} = 18x + 30 \qquad (1.3)$

Put $x = 2$ in (1.3), we get

$$\left(\frac{dm}{dt}\right)_{x=2} = 18(2) + 30 = 36 + 30 = 66$$

Hence the rate of slope of given curve increases 66 units per minute when $x = 2$.

Q.2. If $y = 5 - 3x^2 + 2x^3$ and x decreases at the rate of 6 units per seconds, how fast is the slope of the curve changes when $x = 7$.

Sol. Let t represents the time.

Given that $y = 5 - 3x^2 + 2x^3 \qquad (2.1)$

and $\frac{dx}{dt} = -6 \qquad (2.2)$

Let m be the slope of the curve.

$\therefore \quad m = \frac{dy}{dx}$

$$\Rightarrow \quad m = \frac{d}{dx}\left(5 - 3x^2 + 2x^3\right) \qquad (used\ (2.1))$$

$$\Rightarrow \quad m = -6x + 6x^2$$

Differentiating it with respect to t, we get

$$\frac{dm}{dt} = \frac{d}{dt}\left(-6x + 6x^2\right)$$

$$\Rightarrow \quad \frac{dm}{dt} = (-6 + 12x)\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dm}{dt} = (-6 + 12x).(-6) \qquad (used\ (2.2))$$

$$\Rightarrow \quad \frac{dm}{dt} = 36 - 72x \qquad (2.3)$$

Put $x = 7$ in (2.3), we get

$$\left(\frac{dm}{dt}\right)_{x=7} = 36 - 72(7) = 36 - 504 = -468$$

Hence the rate of slope of given curve decreases 468 units per second when $x = 7$.

Q.3. A particle is moving along a straight line such that the displacement s after time t is given by $s = 2t^2 + t + 7$. Find the velocity and acceleration at time $t = 20$.

Sol. Let v be the velocity and a be the acceleration of the particle at time t.

Given that the displacement of the particle is $s = 2t^2 + t + 7$

Differentiating it with respect to t, we get

$$\frac{ds}{dt} = \frac{d}{dt}\left(2t^2 + t + 7\right)$$

$$\Rightarrow \quad v = 4t + 1 \qquad (3.1)$$

Again differentiating with respect to t, we get

$$\frac{dv}{dt} = \frac{d}{dt}(4t + 1)$$

$$\Rightarrow \quad a = 4 \qquad (3.2)$$

Put $t=20$ in (3.1) and (3.2), we get

$$[v]_{t=20}=4(20)+1=81 \quad \& \quad [a]_{t=20}=4$$

Hence velocity of the particle is 20 and acceleration is 4 when $t=20$.

Q.4. If a particle is moving in a straight line such that the displacement s after time t is given by $s=\frac{1}{2}vt$, where v be the velocity of the particle. Prove that the acceleration a of the particle is constant.

Sol. Given that the displacement of the particle is $s=\frac{1}{2}vt$

Differentiating it with respect to t, we get

$$\frac{ds}{dt}=\frac{d}{dt}\left(\frac{1}{2}vt\right)$$

$$\Rightarrow \quad v=\frac{1}{2}\left[v\frac{dt}{dt}+t\frac{dv}{dt}\right]$$

$$\Rightarrow \quad v=\frac{1}{2}v+\frac{1}{2}t\frac{dv}{dt}$$

$$\Rightarrow \quad v-\frac{1}{2}v=\frac{1}{2}t\frac{dv}{dt}$$

$$\Rightarrow \quad \frac{v}{2}=\frac{1}{2}t\frac{dv}{dt}$$

$$\Rightarrow \quad v=ta$$

Again differentiating with respect to t, we get

$$\frac{dv}{dt}=\frac{d}{dt}(ta)$$

$$\Rightarrow \quad \frac{dv}{dt}=t\frac{da}{dt}+a\frac{dt}{dt}$$

$$\Rightarrow \quad a=t\frac{da}{dt}+a$$

$$\Rightarrow \quad t\frac{da}{dt}=0$$

$\Rightarrow \qquad \frac{da}{dt} = 0$

This shows that acceleration of the particle is always constant.

Q.5. Find the rate of change per second of the area of the circle with respect to its radius r when $r = 4\,cm.$.

Sol. Given that r be the radius of the circle.

Let A be the area of the circle.

$\therefore \qquad A = \pi r^2$

$\Rightarrow \qquad \frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right)$

$\Rightarrow \qquad \frac{dA}{dr} = 2\pi r \qquad (5.1)$

Put $r = 4$ in (5.1), we get

$$\left(\frac{dA}{dr}\right)_{r=4} = 2\pi \times 4 = 8\pi$$

Hence the rate of change of area of the circle is $8\pi\, cm^2 / \sec$.

Q.6. The radius of the circle increases at the rate $0.4\,cm/\sec$. What is the increase of its circumference.

Sol. Let r be the radius and C be the circumference of the circle.

Given that $\frac{dr}{dt} = 0.4\,cm/\sec \qquad (6.1)$

Now $\qquad C = 2\pi r$

$\Rightarrow \qquad \frac{dC}{dt} = \frac{d}{dt}(2\pi r)$

$\Rightarrow \qquad \frac{dC}{dt} = 2\pi \frac{dr}{dt} = 2\pi \times 0.4 = 0.8\pi\, cm/\sec \qquad (used\ (6.1))$

Hence circumference of the circle increases at the rate $0.8\pi\, cm/\sec$.

Q.7. Find the rate of change of the volume of a ball with respect to its radius r.

Sol. Given that r be the radius of the ball.

Let V be the volume of the ball.

$$\therefore \quad V=\frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \frac{dV}{dr}=\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \quad \frac{dV}{dr}=\frac{4}{3}\pi\times 3r^2=4\pi r^2$$

which is the required rate of change of the volume of a ball with respect to its radius r.

Q.8. Find the rate of change of the surface area of a ball with respect to its radius r.

Sol. Given that r be the radius of the ball.

Let S be the surface area of the ball.

$$\therefore \quad S=4\pi r^2$$

$$\Rightarrow \quad \frac{dS}{dr}=\frac{d}{dr}\left(4\pi r^2\right)$$

$$\Rightarrow \quad \frac{dS}{dr}=4\pi\times 2r=8\pi r$$

which is the required rate of change of the surface area of a ball with respect to its radius r.

Q.9. Find the rate of change per second of the volume of a ball with respect to its radius r when $r=6\,cm$.

Sol. Given that r be the radius of the ball.

Let V be the volume of the ball.

$$\therefore \quad V=\frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \frac{dV}{dr}=\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \quad \frac{dV}{dr}=\frac{4}{3}\pi\times 3r^2=4\pi r^2 \qquad (9.1)$$

Put $r=6$ in (9.1), we get

$$\left(\frac{dV}{dr}\right)_{r=6}=4\pi\times(6)^2=144\pi$$

Hence the rate of change of volume of the ball is $144\pi\, cm^3/\sec$.

Q.10. Find the rate of change per minute of the surface area of a ball with respect to its radius r when $r=9\,m$.

Sol. Given that r be the radius of the ball.

Let S be the surface area of the ball.

$$\therefore \quad S=4\pi r^2$$

$$\Rightarrow \quad \frac{dS}{dr}=\frac{d}{dr}\left(4\pi r^2\right)$$

$$\Rightarrow \quad \frac{dS}{dr}=4\pi\times 2r=8\pi r \qquad (10.1)$$

Put $r=9$ in (10.1), we get

$$\left(\frac{dS}{dr}\right)_{r=9}=8\pi\times 9=72\pi$$

Hence the rate of change of surface area of the ball is $72\pi\, m^2/\min$.

Q.11. The radius of an air bubble increases at the rate of $2\,cm/\sec$. At what rate is the volume of the bubble increases when the radius is $5\,cm$?

Sol. Let r be the radius, V be the volume of the bubble and t represents time.

So, by given statement $\frac{dr}{dt}=2\,cm/\sec$ (11.1)

and $V=\frac{4}{3}\pi r^3$

$$\Rightarrow \quad \frac{dV}{dt}=\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \quad \frac{dV}{dt}=\frac{4}{3}\pi\times 3r^2\times\frac{dr}{dt}=4\pi r^2\frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dV}{dt} = 8\pi r^2 \qquad (11.2)$$

$(used\,(11.1))$

Put $r=5$ in (11.2), we get

$$\left(\frac{dV}{dt}\right)_{r=5} = 8\pi \times (5)^2 = 200\pi$$

Hence volume of the bubble increases at the rate $200\pi\, cm^3/\sec$.

Q.12. Find the rate of change of the volume of the cone with respect to the radius of its base.

Sol. Let r be the radius of the base, h be the height and V be the volume of the cone.

$$\therefore \quad V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad \frac{dV}{dr} = \frac{d}{dr}\left(\frac{1}{3}\pi r^2 h\right)$$

$$\Rightarrow \quad \frac{dV}{dr} = \frac{1}{3}\pi h \times 2r = \frac{2}{3}\pi r h.$$

Q.13. Find the rate of change of the volume of the cone with respect to its height.

Sol. Let r be the radius of the base, h be the height and V be the volume of the cone.

$$\therefore \quad V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad \frac{dV}{dh} = \frac{d}{dh}\left(\frac{1}{3}\pi r^2 h\right)$$

$$\Rightarrow \quad \frac{dV}{dh} = \frac{1}{3}\pi r^2.$$

Q.14. Find the rate of change of the surface area of the cone with respect to the radius of its base.

Sol. Let r be the radius of the base, l be the slant length and S be the surface area of the cone.

$$\therefore \quad S = \pi r l + \pi r^2$$

$$\Rightarrow \quad \frac{dS}{dr} = \frac{d}{dr}\left(\pi r l + \pi r^2\right)$$

$$\Rightarrow \quad \frac{dS}{dr} = \pi l + \pi \times 2r$$

$$\Rightarrow \quad \frac{dS}{dr} = \pi l + 2\pi r \text{ .}$$

Q.15. Sand is pouring from a pipe at the rate $10\,cc/\sec$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-fifth of the radius of the base. How fast the height of the sand cone increases when the height is $6\,cm$?

Sol. Let r be the radius of the base, h be the height and V be the volume of the cone at the time t.

So, by given statement $h = \frac{r}{5}$ (15.1)

and $V = \frac{1}{3}\pi r^2 h$

$$\Rightarrow \quad V = \frac{1}{3}\pi(5h)^2 h = \frac{25}{3}\pi h^3 \qquad (used(15.1))$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{d}{dt}\left(\frac{25}{3}\pi h^3\right) = \frac{25}{3}\pi \times 3h^2 \frac{dh}{dt} = 25\pi h^2 \frac{dh}{dt} \qquad (15.2)$$

Also, by given statement $\frac{dV}{dt} = 10\,cc/\sec$ (15.3)

From (15.2) and (15.3), we get

$$25\pi h^2 \frac{dh}{dt} = 10$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{10}{25\pi h^2} = \frac{2}{5\pi h^2}$$

When $h = 6\,cm$, $\frac{dh}{dt} = \frac{2}{5\pi(6)^2} = \frac{1}{90\pi}$

Hence the rate of increases of height of the sand cone is $\frac{1}{90\pi}\,cm/\sec$, when $h = 6\,cm$.

Q.16. The length of edges of a cube increases at the rate of $2\,cm/\sec$. At what rate is the volume of the cube increases when the edge length is $1\,cm$?

Sol. Let a be the length of edge and V be the volume of the cube at time t.

So, by given statement $\frac{da}{dt}=2\,cm/\sec$ (16.1)

and $V=a^3$

$$\Rightarrow \quad \frac{dV}{dt}=\frac{d}{dt}\left(a^3\right)$$

$$\Rightarrow \quad \frac{dV}{dt}=3a^2\frac{da}{dt}$$

$$\Rightarrow \quad \frac{dV}{dt}=6a^2 \qquad (16.2)$$

$(used\,(16.1))$

Put $a=1$ in (16.2), we get

$$\left(\frac{dV}{dt}\right)_{a=1}=6(1)^2=6$$

Hence volume of the cube increases at the rate $6\,cm^3/\sec$.

Q.17. The total revenue received from the sale of x units of a product is given by

$$R(x)=6x^2+18x+20$$

Find the marginal revenue when $x=10$.

Sol. Given that $R(x)=6x^2+18x+20$

Marginal revenue $m(x)$ is given by

$$m(x)=\frac{d}{dx}(R(x))$$

$$\Rightarrow \quad m(x)=\frac{d}{dx}\left(6x^2+18x+20\right)$$

$$\Rightarrow \quad m(x)=12x+18 \qquad (17.1)$$

Put $x=10$ in (17.1), we get

$$m(10)=12(10)+18=138$$

Hence the marginal revenue is 138 when $x=10$.

(b) Maxima and Minima

Maximum Value of a Function & Point of Maxima: Let $f(x)$ be a function defined on domain $D\subset R$. Let a be any point of domain D. We say that $f(x)$ has maximum value at a if $f(x)\le f(a)$ for all $x\in D$ and a is called the point of maxima.

e.g. Let $f(x)=-x^2+5$ for all $x\in R$

Now $x^2\ge 0$ for all $x\in R$

$-x^2\le 0$ for all $x\in R$

$-x^2+5\le 5$ for all $x\in R$

i.e. $f(x)\le 5$ for all $x\in R$

Hence 5 is the maximum value of $f(x)$ which is attained at $x=0$. Therefore $x=0$ is the point of maxima.

Minimum Value of a Function & Point of Minima: Let $f(x)$ be a function defined on domain $D\subset R$. Let a be any point of domain D. We say that $f(x)$ has minimum value at a if $f(x)\ge f(a)$ for all $x\in D$ and a is called the point of minima.

e.g. Let $f(x)=x^2+8$ for all $x\in R$

Now $x^2\ge 0$ for all $x\in R$

$x^2+8\ge 8$ for all $x\in R$

i.e. $f(x)\ge 8$ for all $x\in R$

Hence 8 is the minimum value of $f(x)$ which is attained at $x=0$. Therefore $x=0$ is the point of minima.

Note: We can also attain points of maxima and minima & their corresponding maximum and minimum value of a given function by differential calculus too.

Working Rule to find points of maxima or minima or inflexion by Differential Calculus:

Step No.	Working Procedure
1	Put $y = f(x)$
2	Find $\frac{dy}{dx}$
3	Put $\frac{dy}{dx}=0$ and solve it for x. Let x_1 , x_2 , . . . , x_n are the values of x.
4	Find $\frac{d^2y}{dx^2}$.
5	Put the values of x in $\frac{d^2y}{dx^2}$. Suppose $x=x_i$ be any value of x. If $\frac{d^2y}{dx^2}<0$ at $x=x_i$ then $x=x_i$ is the point of maxima and $f(x_i)$ is the maximum value of $f(x)$. If $\frac{d^2y}{dx^2}>0$ at $x=x_i$ then $x=x_i$ is the point of minima and $f(x_i)$ is minimum value of $f(x)$. If $\frac{d^2y}{dx^2}=0$ at $x=x_i$. Find $\frac{d^3y}{dx^3}$. If $\frac{d^3y}{dx^3}\neq 0$ at $x=x_i$ then $x=x_i$ is the point of inflexion.

Questions of Maxima and Minima:

Q.1. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x)=x^3-12x^2+5$.

Sol. Let $y=f(x)=x^3-12x^2+5$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^3-12x^2+5\right)$$

$$\frac{dy}{dx}=3x^2-24x$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(3x^2-24x\right)$$

$$\frac{d^2 y}{dx^2} = 6x - 24$$

Put $\frac{dy}{dx} = 0$, we get

$$3x^2 - 24x = 0$$

$$3x(x-8) = 0$$

$$Either\ \ x = 0\ \ or\ \ x - 8 = 0$$

$$Either\ \ x = 0\ \ or\ \ x = 8$$

When $x = 0$:

$$\left(\frac{d^2 y}{dx^2}\right)_{x=0} = (6x - 24)_{x=0} = -24 < 0$$

which shows that $x = 0$ is a point of maxima.

So maximum value of $f(x) = x^3 - 12x^2 + 5$ is

$$(y)_{x=0} = \left(x^3 - 12x^2 + 5\right)_{x=0} = 5$$

When $x = 8$:

$$\left(\frac{d^2 y}{dx^2}\right)_{x=8} = (6x - 24)_{x=8} = 6(8) - 24 = 24 > 0$$

which shows that $x = 8$ is a point of minima.

So minimum value of $f(x) = x^3 - 12x^2 + 5$ is

$$(y)_{x=8} = \left(x^3 - 12x^2 + 5\right)_{x=8} = 8^3 - 12(8)^2 + 5$$

$$= 512 - 768 + 5 = -251$$

Q.2. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = 6x^3 - 27x^2 + 36x + 6$.

Sol. Let $y = f(x) = 6x^3 - 27x^2 + 36x + 6$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(6x^3-27x^2+36x+6\right)$$

$$\frac{dy}{dx}=18x^2-54x+36$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(18x^2-54x+36\right)$$

$$\frac{d^2y}{dx^2}=36x-54$$

Put $\frac{dy}{dx}=0$, we get

$$18x^2-54x+36=0$$

$$x^2-3x+2=0$$

$$x^2-2x-x+2=0$$

$$x(x-2)-1(x-2)=0$$

$$(x-1)(x-2)=0$$

$$Either\ \ x-1=0\ \ or\ \ x-2=0$$

$$Either\ \ x=1\ \ or\ \ x=2$$

When $x=1$:

$$\left(\frac{d^2y}{dx^2}\right)_{x=1}=(36x-54)_{x=1}=36-54=-18<0$$

which shows that $x=1$ is a point of maxima.

So maximum value of $f(x)=6x^3-27x^2+36x+6$ is

$$(y)_{x=1}=\left(6x^3-27x^2+36x+6\right)_{x=1}=6-27+36+6=21$$

When $x=2$:

$$\left(\frac{d^2 y}{d x^2}\right)_{x=2} = (36x-54)_{x=2} = 72-54 = 18 > 0$$

which shows that $x=2$ is a point of minima.

So minimum value of $f(x)=6x^3-27x^2+36x+6$ is

$$(y)_{x=2} = (6x^3-27x^2+36x+6)_{x=2} = 48-108+72+6 = 18$$

Q.3. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x)=-2x^3+6x^2+18x-1$.

Sol. Let $y=f(x)=-2x^3+6x^2+18x-1$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(-2x^3+6x^2+18x-1\right)$$

$$\frac{dy}{dx}=-6x^2+12x+18$$

Again differentiating with respect to x, we get

$$\frac{d^2 y}{d x^2}=\frac{d}{dx}\left(-6x^2+12x+18\right)$$

$$\frac{d^2 y}{d x^2}=-12x+12$$

Put $\frac{dy}{dx}=0$, we get

$$-6x^2+12x+18=0$$

$$-6\left(x^2-2x-3\right)=0$$

$$x^2-2x-3=0$$

$$x^2-3x+x-3=0$$

$$x(x-3)+1(x-3)=0$$

$$(x+1)(x-3)=0$$

$$Either\ \ x+1=0\ \ or\ \ \ x-3=0$$

$$Either\ \ x=-1\ \ or\ \ \ x=3$$

When $x=-1$:

$$\left(\frac{d^2y}{dx^2}\right)_{x=-1}=(-12x+12)_{x=-1}=12+12=24>0$$

which shows that $x=-1$ is a point of minima.

So minimum value of $f(x)=-2x^3+6x^2+18x-1$ is

$$(y)_{x=-1}=\left(-2x^3+6x^2+18x-1\right)_{x=-1}$$

$$=-2(-1)^3+6(-1)^2+18(-1)-1$$

$$=2+6-18-1=-11$$

When $x=3$:

$$\left(\frac{d^2y}{dx^2}\right)_{x=3}=(-12x+12)_{x=3}=-36+12=-24<0$$

which shows that $x=3$ is a point of maxima.

So maximum value of $f(x)=-2x^3+6x^2+18x-1$ is

$$(y)_{x=3}=\left(-2x^3+6x^2+18x-1\right)_{x=3}$$

$$=-2(3)^3+6(3)^2+18(3)-1$$

$$=-54+54+54-1=53$$

Q.4. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x)=\sin x+\cos x\ \ where\ \ 0\leq x\leq\frac{\pi}{2}$.

Sol. Let $y=f(x)=\sin x+\cos x$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\sin x+\cos x)$$

$$\frac{dy}{dx}=\cos x-\sin x$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(\cos x-\sin x)$$

$$\frac{d^2y}{dx^2}=-\sin x-\cos x$$

Put $\frac{dy}{dx}=0$, we get

$$\cos x-\sin x=0$$

$$\cos x=\sin x$$

$$\frac{\cos x}{\sin x}=1$$

$$\cot x=1$$

$$\cot x=\cot\frac{\pi}{4} \qquad as\ 0\le x\le\frac{\pi}{2}$$

$$x=\frac{\pi}{4}$$

When $x=\frac{\pi}{4}$:

$$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{\pi}{4}}=(-\sin x-\cos x)_{x=\frac{\pi}{4}}=-\sin\frac{\pi}{4}-\cos\frac{\pi}{4}$$

$$=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=-\frac{2}{\sqrt{2}}<0$$

which shows that $x=\frac{\pi}{4}$ is a point of maxima.

So maximum value of $f(x)=\sin x+\cos x$ is

$$(y)_{x=\frac{\pi}{4}}=(\sin x+\cos x)_{x=\frac{\pi}{4}}$$

$$=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}$$

Q.5. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x)=\frac{\log x}{x}$ *if* $0<x<\infty$.

Sol. Let $y=f(x)=\frac{\log x}{x}$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\log x}{x}\right)$$

$$\frac{dy}{dx}=\frac{x\frac{d}{dx}(\log x)-\log x\frac{dx}{dx}}{x^2}$$

$$\frac{dy}{dx}=\frac{x\times\frac{1}{x}-\log x}{x^2}$$

$$\frac{dy}{dx}=\frac{1-\log x}{x^2}$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1-\log x}{x^2}\right)$$

$$\frac{d^2y}{dx^2}=\frac{x^2\frac{d}{dx}(1-\log x)-(1-\log x)\frac{d}{dx}(x^2)}{x^4}$$

$$\frac{d^2y}{dx^2}=\frac{x^2\left(0-\frac{1}{x}\right)-(1-\log x)(2x)}{x^4}$$

$$\frac{d^2y}{dx^2}=\frac{-x-2x+2x\log x}{x^4}=\frac{-1-2+2\log x}{x^3}=\frac{-3+2\log x}{x^3}$$

Put $\frac{dy}{dx}=0$, we get

$$\frac{1-\log x}{x^2}=0$$

$$1-\log x=0$$

$$\log x=1$$

$$x=e$$

When $x=e$:

$$\left(\frac{d^2y}{dx^2}\right)_{x=e}=\left(\frac{-3+2\log x}{x^3}\right)_{x=e}=\frac{-3+2\log e}{e^3}$$

$$=\frac{-3+2}{e^3}=\frac{-1}{e^3}<0 \qquad (used\ \log e=1)$$

which shows that $x=e$ is a point of maxima.

So maximum value of $f(x)=\frac{\log x}{x}$ is

$$(y)_{x=e}=\left(\frac{\log x}{x}\right)_{x=e}=\frac{\log e}{e}=\frac{1}{e}$$

Q.6. Find two positive numbers x & y such that $x.y=16$ and the sum $x+y$ is minimum. Also find the minimum value of sum.

Sol. Given that $x.y=16$ $\Rightarrow y=\frac{16}{x}$

Let $S=x+y$ $\Rightarrow S=x+\frac{16}{x}$

Differentiating it with respect to x, we get

$$\frac{dS}{dx}=\frac{d}{dx}\left(x+\frac{16}{x}\right)$$

$$\frac{dS}{dx}=1+16\left(-\frac{1}{x^2}\right)=1-\frac{16}{x^2}$$

Again differentiating with respect to x, we get

$$\frac{d^2 S}{d x^2}=\frac{d}{d x}\left(1-\frac{16}{x^2}\right)$$

$$\frac{d^2 S}{d x^2}=0-16\left(-\frac{2}{x^3}\right)=\frac{32}{x^3}$$

Put $\frac{d S}{d x}=0$, we get

$$1-\frac{16}{x^2}=0$$

$$\frac{x^2-16}{x^2}=0$$

$$x^2-16=0$$

$$Either \quad x=4 \quad or \quad x=-4$$

$x=-4$ is rejected as x is positive.

When $x=4$:

$$\left(\frac{d^2 S}{d x^2}\right)_{x=4}=\left(\frac{32}{x^3}\right)_{x=4}=\frac{32}{4^3}=\frac{32}{64}=\frac{1}{2}>0$$

which shows that $x=4$ is a point of minima.

Now at $x=4$, value of y is :

$$(y)_{x=4}=\left(\frac{16}{x}\right)_{x=4}=\frac{16}{4}=4$$

Also minimum value of sum $S=x+y$ is

$$(S)_{x=4,\,y=4}=(x+y)_{x=4,\,y=4}=4+4=8$$

Q.7. Find the dimensions of the rectangle of given area 169 sq. c.m. whose perimeter is least. Also find its perimeter.

Sol. Let the sides of the rectangle are $x \text{ and } y$, A be the area and P be the perimeter.

$$\therefore \qquad A=x\,y=169\,sq.c.m. \qquad \Rightarrow \; y=\frac{169}{x}$$

And $P=2(x+y)$ $\Rightarrow P=2\left(x+\frac{169}{x}\right)=2x+\frac{338}{x}$

Differentiating it with respect to x, we get

$$\frac{dP}{dx}=\frac{d}{dx}\left(2x+\frac{338}{x}\right)$$

$$\frac{dP}{dx}=2+338\left(-\frac{1}{x^2}\right)=2-\frac{338}{x^2}$$

Again differentiating with respect to x, we get

$$\frac{d^2P}{dx^2}=\frac{d}{dx}\left(2-\frac{338}{x^2}\right)$$

$$\frac{d^2P}{dx^2}=0-338\left(-\frac{2}{x^3}\right)=\frac{676}{x^3}$$

Put $\frac{dP}{dx}=0$, we get

$$2-\frac{338}{x^2}=0$$

$$\frac{2x^2-338}{x^2}=0$$

$$2x^2-338=0$$

$$x^2=169$$

$Either\quad x=13\quad or\quad x=-13$

$x=-13$ is rejected as x can't be negative.

When $x=13$:

$$\left(\frac{d^2P}{dx^2}\right)_{x=13}=\left(\frac{676}{x^3}\right)_{x=13}=\frac{676}{(13)^3}=\frac{4}{13}>0$$

which shows that $x=13$ is a point of minima.

Therefore, Perimeter is least at $x=13$.

Now at $x=13$, value of y is :

$$(y)_{x=13}=\left(\frac{169}{x}\right)_{x=13}=\frac{169}{13}=13$$

Also least value of perimeter $P=2(x+y)$ is

$$(P)_{x=13,\,y=13}=(2x+2y)_{x=13,\,y=13}=26+26=52\,c.m.$$

Q.8. Show that among all the rectangles of a given perimeter, the square has the maximum area.

Sol. Let the sides of the rectangle are $x \text{ and } y$, A be the area and P be the given perimeter.

$$\therefore \quad P=2(x+y) \quad \Rightarrow P=2x+2y \quad \Rightarrow y=\frac{P-2x}{2}=\frac{P}{2}-x$$

And

$$A=xy=\frac{Px}{2}-x^2$$

Differentiating it with respect to x, we get

$$\frac{dA}{dx}=\frac{d}{dx}\left(\frac{Px}{2}-x^2\right)$$

$$\frac{dA}{dx}=\frac{P}{2}-2x$$

Again differentiating with respect to x, we get

$$\frac{d^2A}{dx^2}=\frac{d}{dx}\left(\frac{P}{2}-2x\right)$$

$$\frac{d^2A}{dx^2}=0-2=-2$$

Put $\frac{dA}{dx}=0$, we get

$$\frac{P}{2}-2x=0$$

$$x=\frac{P}{4}$$

When $x=\dfrac{P}{4}$:

$$\left(\frac{d^2 A}{d x^2}\right)_{x=\frac{P}{4}} = -2 < 0$$

which shows that $x=\dfrac{P}{4}$ is a point of maxima.

Therefore, Area is maximum at $x=\dfrac{P}{4}$.

Now at $x=\dfrac{P}{4}$, value of y is :

$$(y)_{x=\frac{P}{4}} = \left(\frac{P}{2}-x\right)_{x=\frac{P}{4}} = \frac{P}{2}-\frac{P}{4}=\frac{P}{4}$$

$\Rightarrow$ $x=y=\dfrac{P}{4}$ gives the maximum area.

Hence among all the rectangles of a given perimeter, the square has the maximum area.

Q.9. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x)=x^3+1$.

Sol. Let $y=f(x)=x^3+1$

Differentiating it with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^3+1\right)$$

$$\frac{dy}{dx}=3x^2$$

Again differentiating with respect to x, we get

$$\frac{d^2 y}{dx^2}=\frac{d}{dx}\left(3x^2\right)$$

$$\frac{d^2 y}{dx^2}=6x$$

Put $\frac{dy}{dx}=0$, we get

$$3x^2=0$$

$$\Rightarrow \quad x=0$$

When $x=0$:

$$\left(\frac{d^2y}{dx^2}\right)_{x=0}=(6x)_{x=0}=6\times 0=0$$

To check maxima or minima, we need to find third order derivative of y with respect to x.

So, $$\frac{d^3y}{dx^3}=\frac{d}{dx}(6x)$$

$$\Rightarrow \quad \frac{d^3y}{dx^3}=6\neq 0$$

which shows that $x=0$ is neither a point of maxima nor a point of minima, hence the given function has neither maximum value nor minimum value.